# THE SPATIAL $\Lambda$-FLEMING-VIOT PROCESS IN A RANDOM ENVIRONMENT

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## Contents



## Introduction

The main aim of this paper is to establish a connection between to class of processes in static random potential. The first one belongs to a family of spatial Lambda-Fleming-Viot processes, which describes evolution of frequencies of genetic types in the spatially distributed population. The second one is the rough superBrownian motion. We are motivated by a question of biological interest: does spatially heterogeneous selection enhance the genetic diversity?

Just as the superBrownian motion, introduced independently by [11] and [34] is a continuous in time and space approximation of a system of critical branching random walks on a $d$-dimensional lattice, the rough superBrownian motion, introduced and studied by [25] is an approximation of the branching random walk in a static random potential. Both processes may be understood as a spatial counterparts of Feller diffusion.

The difficulties of modelling evolution of spatially distributed populations are well know, and often referred to as 'pain in the torus' (see [5]). A modelling approach which overcomes most of them has been found in the class of measure valued models called spatial Lambda-Fleming-Viot processes, introduced in [13], [4]. This approach can be seen as an extension of the classical Wright-Fisher or Cannings models to the spatial continuum, i.e. the non-spatial variants of Lambda-Fleming-Viot models and Wright-Fisher models yield the same diffusion approximations.

The connection between the spatial Lambda-Fleming-Viot model in random static environment and rough superBrownian motion is not entirely surprising. It is well known that dynamics of the small subpopulation in the Wright-Fisher model is described by the branching process, whose diffusion approximation is Feller diffusion. The superBrownian motion (with both finite and infinite variance) has been recovered from the 'neutral' variant of the spatial

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Lambda-Fleming-Viot model as a scaling limit in any dimension by [8]. The finite variance superBrownian motion, under different, 'critical' scaling regime, has been also recovered in dimension 2 by [10], again from the neutral variant of the model. Both results rely on stochastic duality techniques. The superBrownian motion in a random environment has been recovered from the spatial lambda-Fleming-Viot model with selection in fluctuating environment by [9], with a technique based on particle representation.

All of aforementioned results relied on probabilistic methods, since the noises under consideration have been white in time as well, which made the martingale techniques available. The presence of static random potential means that those techniques are replaced by others which are more analytical in flavour. Our analysis is based on a similarity of the sequence of approximating processes conditioned on the realisation of random environments to Parabolic Anderson Model. In particular, the continuous variant of this model in dimension $d = 2$ requires special probabilistic techniques in the spirit of rough paths. In our work we have chosen to follow the approach of para-controlled distributions, see i.e. [15].

From the modelling perspective, it is interesting to observe that the rough superBrownian motion, even on a torus, is persistent in dimensions $d = 1, 2$. This is in the sharp contrast to classical superBrownian motion which is suffers weak local extinction in those dimensions, even in the full space.

**Biological background.** Since the seminal paper [35] it has been understood that one of the sources of genetic variation comes from genetic isolation. Since individuals inhabit different, possibly distant geographical regions and, at least for most of the species, do not move too far from their place of birth, the likelihood of mating between geographically distant populations is very small. This leads to a greater differentiation between subpopulations, as distant individuals evolve essentially independently of each other. In extreme cases, this mechanism, which is usually referred to as isolation by distance may even lead to creation of different species.

In principle, selection acts to reduce the genetic variety. However, [35] has speculated that if the selection is heterogeneous, that is, if selection favors different types of individuals in different regions in space, it may further enhance the differentiation coming from isolation by distance.

A large body of empirical evidence suggests that this may indeed be the case. Studies on plants [23], bacteria [26], animals [21] seem seem to all confirm that the spatial environmental heterogeneity enhance the diversity. For more in-depth description of biological literature, including less favourable viewpoints of the phenomena we are concerned with, we refer to [30], [17], [29].

One of the indirect methods to determine whether a certain set of environmental conditions may enhance genetic variety is to try to determine the fate of an establishing mutation. If a new mutation is favoured in all regions of geographical space and it manages to survive the initial demographic stochasticity (genetic drift), selection will lead to a population which consists only of individuals of the new, favoured type. If the selection is heterogeneous, that is when we observe regions in which selection acts at opposite directions, the picture is much less clear.

As discussed before, the dynamics of establishing mutation in neutral, spatially distributed population can be modelled by a superBrownian motion. In dimensions $d = 1, 2$ this process is not persistent, which means that the new mutation will be lost in the process of evolution with probability one. By showing that the analogous population subject to heterogeneous, static selection persists, we provide a weak circumstantial evidence for Wright's claim.

**Structure of the paper.** The rest of the paper is laid out as follows. In Section 1, we describe the notations, define the models and state main results. Section 2 is devoted to relation between Spatial Lambda-Fleming-Viot process with selection in rough potential and rough superBrownian motion, whereas Section 3 the similar relationship with Fisher-KPP equation in rough potential. Questions of persistence and long time behaviour of the process are addressed as well. The rest of the paper is devoted to analytical and stochastic backbone of our results. Namely, in Section **??** we recall some results on function spaces and paracontrolled calculus. Section **??** covers Schauder estimates. Finally, Section 6 is devoted to stochastic estimates which allow us to characterize the noise.

## 1. Models and statement of main results

We begin with stating the notation used throughout the paper in Subsection 1.1. In Subsection 1.2 we describe the Spatial-Lambda-Fleming-Viot. In Subsection 1.3 describe the small famillies limit which is the first of our main results. In Subsection 1.4 scaling limit to FKPP.

### 1.1. Notations.

We write $\mathbb{N} = \{0, 1, 2, \dots\}$, and $\mathbb{R}_+ = [0, \infty)$. Fix $d \in \mathbb{N}$. We define $d-$dimensional torus $\mathbb{T}^d = [-1/2, 1/2]^d /\!\sim$, where $\sim$ is the equivalence relation which glues opposite edges. For notational convenience we use the following convention.

$$\text{We write} \quad \varepsilon \in (0, 1/2) \quad \text{for} \quad \varepsilon = \frac{1}{n}, \quad \text{for some} \quad n \in \mathbb{N}, n \geq 2.$$

Then we introduce the following scaling for balls and cubes. Indicate with $|A|$ the Lebesgue measure of a Borel set $A \subseteq \mathbb{T}^d$. Let then $B_\varepsilon(x) \subseteq \mathbb{T}^d$ be the ball (w.r.t. the Euclidian norm) of volume $\varepsilon^d$ about $x$. Similarly, let $Q_\varepsilon(x) \subset \mathbb{T}^d$ be the $d$-dimensional cube

$$y \in Q_\varepsilon(x) \iff (y - x)_i \in \Big[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big), \ \forall i \in \{1, \dots, d\}.$$

In particular, in our notation

$$|B_\varepsilon(x)| = |Q_\varepsilon(x)| = \varepsilon^d.$$

Now, for integrable $w \colon \mathbb{T}^d \to \mathbb{R}$ define $\Pi_\varepsilon w(x)$ as an average integral of $w$ over $B_\varepsilon(x)$, that is

$$\Pi_\varepsilon w(x) := \fint_{B_\varepsilon(x)} w(y) \, \mathrm{d}y := \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} w(y) \, \mathrm{d}y.$$

Furthermore, consider the lattice

$$\mathbb{Z}^d_\varepsilon = \Big(\varepsilon^{-1} \mathbb{Z}^d\Big) \cap \mathbb{T}^d.$$

Since $\varepsilon = \frac{1}{n}$, cubes $Q_\varepsilon$, centred at the points of lattice $\mathbb{Z}^d_\varepsilon$ are disjoint and satisfy:

$$\mathbb{T}^d = \bigcup_{x \in \mathbb{Z}_\varepsilon} Q_\varepsilon(x).$$

It will be useful to consider the Fourier transform both on the torus and in the full space. For $\varphi \in \mathcal{S}'(\mathbb{T}^d)$, that is for an element of the space of tempered distributions on $\mathbb{T}^d$, we define

$$\widehat{\varphi}(k) = \mathcal{F}_{\mathbb{T}^d} \varphi(k) = \int_{\mathbb{T}^d} e^{-2\pi \iota k \cdot x} f(x) \, \mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Analogously, for $\psi \in \mathcal{S}'(\mathbb{R}^d)$

$$\mathcal{F}_{\mathbb{R}^d} \psi(k) = \int_{\mathbb{R}^d} e^{-2\pi \iota k \cdot x} \psi(x) \, \mathrm{d}x, \quad k \in \mathbb{R}^d.$$

These Fourier transforms admit inverses, which we denote with $\mathcal{F}^{-1}_{\mathbb{T}^d}, \mathcal{F}^{-1}_{\mathbb{R}^d}$ respectively.

For $a \colon \mathbb{Z}^d \to \mathbb{R}$ with at most polynomial growth we define the Fourier multiplier as an operator of the form

$$a(D)\varphi := \mathcal{F}^{-1}_{\mathbb{T}^d}\big(a(\cdot)\mathcal{F}_{\mathbb{T}^d}\varphi(\cdot)\big), \qquad \forall \varphi \in \mathcal{S}'(\mathbb{T}^d).$$

Since characteristic functions, normalized to integrate to 1 over the entire domain, enter the calculations repeatedly, for a set $A$ we write:

$$\chi_A(x) = \frac{1}{|A|} 1_A(x).$$

In the special case of balls and cubes we additionally define

$$\chi_\varepsilon(x) := \varepsilon^{-d} 1_{B_\varepsilon(0)}(x), \qquad \widehat{\chi}_\varepsilon(k) = \widehat{\chi}(\varepsilon k) := \mathcal{F}_{\mathbb{T}^d}\chi_\varepsilon(k) = \mathcal{F}_{\mathbb{R}^d}\chi_\varepsilon(k),$$
$$\chi_{Q_\varepsilon}(x) := \varepsilon^{-d} 1_{Q_\varepsilon(0)}(x), \qquad \widehat{\chi}_{Q_\varepsilon}(k) = \widehat{\chi}_Q(\varepsilon k) := \mathcal{F}_{\mathbb{T}^d}\chi_{Q_\varepsilon}(k) = \mathcal{F}_{\mathbb{R}^d}\chi_{Q_\varepsilon}(k).$$

Observe that in order to obtain the identity between the Fourier transform on the torus and in the full space, the $\varepsilon$ should satisfy $\varepsilon \leq 1/2 < \sqrt{\pi}/2$, as otherwise the ball of radius $\varepsilon$ about 0 intersects the boundary of the torus.

A notable example for all the above definitions is the operator $\mathcal{A}_\varepsilon$ defined by:

$$\mathcal{A}_\varepsilon(\varphi)(x) = \varepsilon^{-2} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(y)} \varphi(y) - \varphi(x) \, \mathrm{d}z \, \mathrm{d}y = \varepsilon^{-2}\big(\Pi_\varepsilon^2 \varphi - \varphi\big)(x).$$

Such operator is a Fourier multiplier with

$$\mathcal{A}_\varepsilon = \vartheta_\varepsilon(D), \qquad \vartheta_\varepsilon(k) = \frac{1}{\varepsilon^2}\frac{1}{\hat{\chi}^2(\varepsilon k) - 1}.$$

We proceed with a definition of Besov spaces. Following [3, Proposition 2.10], fix a dyadic partition of the unity $\{\varrho_j\}_{j \geq -1}$ where for $j \geq 0$, $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a radial, smooth compactly supported $\varrho$. For a distribution $\varphi \in \mathcal{S}'(\mathbb{T}^d)$ define $\Delta_j \varphi = \varrho_j(D)\varphi$ and hence define the spaces $B^\alpha_{p,q}$ for $\alpha \in \mathbb{T}, p, q \in [1, \infty]$ via the norms

$$\|\varphi\|_{B^\alpha_{p,q}} = \|(2^{\alpha j}\|\Delta_j \varphi\|_{L^p(\mathbb{T}^d)})_{j \geq -1}\|_{\ell^q(j \geq -1)}.$$

Since the partition of unity was chosen to be smooth, we define the Besov spaces on full space via the same formula. It is convenient to introduce notation

$$K^x_j(y) = \mathcal{F}_{\mathbb{T}^d}\rho_j(x - y).$$

For $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$ and $p, q = \infty$ the above definition coincides with that of classical Hölder spaces. We therefore write

$$\mathcal{C}^\alpha := B^\alpha_{\infty,\infty}, \qquad \mathcal{C}^\alpha_p := B^\alpha_{p,\infty}.$$

We shall denote the norm of the Hölder space $\mathcal{C}^\alpha$ by $\|\cdot\|_\alpha$.

Let $\mathcal{M}(\mathbb{T}^d)$ denote the space of finite positive measures over $\mathbb{T}^d$. For metric spaces $X, Y$ let $C(X;Y)$ and $C_b(X;Y)$ the space of continuous, and bounded and continuous, respectively functions from $X$ to $Y$. If $Y = \mathbb{R}$, we may drop the second argument. In addition for a metric space $X$ we define $\mathbb{D}([0,\infty);X)$ to be the space of cadlag functions with values in $X$, endowed with the Skorohod topology as in [14, Section 3.5] (similarly for finite time horizon $T > 0$ we write $\mathbb{D}([0,T];X)$). If $X$ is a Banach space we write $L^2([0,T];X)$ for measurable functions $\varphi$ on $[0,T]$ taking values in $X$ and satisfying $\|\varphi\|_{L^2([0,T];X)} = \big(\int_0^T \|\varphi(s)\|_X^2 \, \mathrm{d}s\big)^{\frac{1}{2}} < \infty$. Then let $L^2_{\mathrm{loc}}([0,\infty);X) = \bigcap_{T>0} L^2([0,T];X)$.

1.2. **Spatial $\Lambda$-Fleming-Viot process in a random environment.** At the center of our attention lies a family $X^\varepsilon$ of Markov processes, indexed by $\varepsilon \in (0, \frac{1}{2})$. The dynamics of this process will depend on the realization of a random environment, which takes the form of a spatial selection coefficient:

(1) $\qquad \Omega \ni \omega \mapsto s_\varepsilon(\omega) \in L^\infty(\mathbb{T}^d;\mathbb{R}), \quad |s_\varepsilon(\omega, x)| < 1, \quad$ with $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

Once the random selection coefficient is fixed, the process $X^\varepsilon$ follows the (again random) dynamics of a spatial $\Lambda$-Fleming-Viot process with selection $s_\varepsilon(\omega)$. Such process describes the spatial distribution of two species, which can for instance be of type $\mathfrak{a}$ or $\mathfrak{A}$. The process $X^\varepsilon$ represents then the fraction of type $\mathfrak{a}$ individuals with respect to the total population. In simple terms:

$$X^\varepsilon_t(\omega, x) = \frac{\#\{\text{Individuals of type } \mathfrak{a} \text{ at time } t \text{ and position } x\}}{\#\{\text{Total pupulation size at time } t \text{ and position } x\}} \in [0,1].$$

The dependence on $\omega$ indicates that the process evolves under the environment $s_\varepsilon(\omega)$. The dynamics of the process $X^\varepsilon$ in the spatial Fleming-Viot models are described by the occurrence of reproduction events. Since we consider also the presence of selection, these reproduction events can be of two kinds, that we again describe in simple terms:

**Neutral:** Both types have the same chance of reproducing,

**Selective:** One of the two types is more likely to reproduce than the other.

More precisely, these reproduction events are distributed according to space-time Poisson point processes (depending on $s_\varepsilon(\omega)$), and the reproduction impacts areas of size $\varepsilon$ around such points. A precise, yet informal, description is provided below. A rigorous construction is deferred to the appendix, add reference.

**Definition 1.1** (Spatial Λ-Fleming-Viot process with random selection)**.** *Fix $\varepsilon \in (0, \frac{1}{2})$, $\mathfrak{u} \in (0,1)$ and consider $s_\varepsilon$ and $\Omega$ as in (1). Let $X^{\varepsilon,0} \colon \mathbb{T}^d \to \mathbb{R}$ be such that $0 \le X^{\varepsilon,0} \le 1$. Define the process $X^\varepsilon$ on the probability space $(\Omega \times \Omega', \widetilde{\mathcal{F}}, \mathbb{P} \ltimes \mathbb{P}^\omega)$ (see add for a precise construction of this space), so that for every $\omega \in \Omega$ it holds that:*

- *The space $(\Omega', \mathbb{P}^\omega)$ supports a pair of independent Poisson point processes $\Pi^{\mathrm{neu}}_\omega$, $\Pi^{\mathrm{sel}}_\omega$ on $\mathbb{R}_+ \times \mathbb{T}^d$ with intensity measures $\mathrm{d}t \otimes (1 - |s_\varepsilon(\omega, x)|)\mathrm{d}x$ and $\mathrm{d}t \otimes |s_\varepsilon(\omega, x)|\mathrm{d}x$ respectively.*
- *The random process (defined on $\Omega'$) $\mathbb{R}_+ \ni t \mapsto X^\varepsilon_t(\omega)$ is a Markov process with values in $L^\infty(\mathbb{T}^d)$ associated to the generator $\mathcal{L}(\varepsilon, s_\varepsilon(\omega), \mathfrak{u})$ (see Definition A.1) started in $X^{\varepsilon,0}$ and described by the following dynamics.*

(1) *If $(t,x) \in \Pi^{\mathrm{neu}}_\omega$, a neutral event occurs at time $t$ in the ball $B_\varepsilon(x)$, namely:*
  (a) *Choose a parental location $y$ uniformly in $B_\varepsilon(x)$.*
  (b) *Choose the parental type $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$ according to the distribution*
$$\mathbb{P}[\mathfrak{p} = \mathfrak{a}] = X^\varepsilon_{t-}(\omega, y), \quad \mathbb{P}[\mathfrak{p} = \mathfrak{A}] = 1 - X^\varepsilon_{t-}(\omega, y).$$
  (c) *A proportion $\mathfrak{u}$ of the population within $B_\varepsilon(x)$ dies and is replaced by offspring with type $\mathfrak{p}$. Therefore, for each point $z \in B(x,r)$,*
$$X^\varepsilon_t(\omega, z) = X^\varepsilon_{t-}(\omega, z)(1 - \mathfrak{u}) + \mathfrak{u}\chi_{\{\mathfrak{p}=\mathfrak{a}\}}.$$

(2) *If $(t,x) \in \Pi^{\mathrm{sel}}_\omega$, a selective event occurs in the ball $B_\varepsilon(x)$, namely:*
  (a) *Choose two parental locations $y_0, y_1$ independently, uniformly in $B_\varepsilon(x)$.*
  (b) *Choose the two parental types, $\mathfrak{p}_0, \mathfrak{p}_1$, independently, according to*
$$\mathbb{P}[\mathfrak{p}_i = \mathfrak{a}] = X^\varepsilon_{t-}(\omega, y_i), \quad \mathbb{P}[\mathfrak{p}_i = \mathfrak{A}] = 1 - X^\varepsilon_{t-}(\omega, y_i).$$
  (c) *A proportion $\mathfrak{u}$ of the population within $B_\varepsilon(x)$ dies and is replaced by offspring with type chosen as follows:*
    (i) *If $s(x) > 0$, their type is set to be a if $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$, and $\mathfrak{A}$ otherwise. Thus for each $z \in B_\varepsilon(x)$*
$$X^\varepsilon_t(\omega, x) = (1 - \mathfrak{u})X^\varepsilon_{t-}(\omega, z) + \mathfrak{u}\chi_{\{\mathfrak{p}_0=\mathfrak{p}_1=\mathfrak{a}\}}.$$
    (ii) *If $s(x) < 0$, their type is set to be $\mathfrak{a}$ if $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$ or $\mathfrak{p}_0 \neq \mathfrak{p}_1$ and $\mathfrak{A}$ otherwise, so that for each $z \in B_\varepsilon(x)$,*
$$X^\varepsilon_t(\omega, z) = (1 - \mathfrak{u})X^\varepsilon_{t-}(\omega, z) + \mathfrak{u}(\chi_{\{\mathfrak{p}_0=\mathfrak{p}_1=\mathfrak{a}\}} + \chi_{\{\mathfrak{p}_0\neq\mathfrak{p}_1\}}).$$

Although in this section we skip the rigorous definition of the generator, in the upcoming discussion the following martingale problem for one-dimensional projections of the process plays a key role. For simplicity write for a function $f$:
$$f_{t,s} = f_t - f_s.$$

**Lemma 1.2.** *Fix $\omega \in \Omega$. For every $\varphi \in C(\mathbb{T}^d)$ the process $t \mapsto \langle X^\varepsilon_t(\omega), \varphi\rangle$ satisfies the following martingale problem, for every $t \ge s \ge 0$:*
$$\langle X^\varepsilon_{t,s}(\omega), \varphi\rangle = \mathfrak{u}\varepsilon^d \int_s^t \langle (\Pi^2_\varepsilon - \mathrm{Id})(X^\varepsilon_r(\omega)), \varphi\rangle + \langle \Pi_\varepsilon\big[s_\varepsilon(\omega)\big(\Pi_\varepsilon X^\varepsilon_r(\omega) - (\Pi_\varepsilon X^\varepsilon_r(\omega))^2\big)\big], \varphi\rangle \,\mathrm{d}r + M^\varepsilon_{t,s}(\varphi)$$

*where $M^{\varepsilon}_{t,s}(\varphi)$ is the increment of a square integrable martingale with predictable quadratic variation given by:*

$$\langle M^{\varepsilon}(\varphi)\rangle_t = \mathfrak{u}^2\varepsilon^{2d}\int_0^t \langle (1+s_{\varepsilon}(\omega))\Pi_{\varepsilon}X^{\varepsilon}_r(\omega), (\Pi_{\varepsilon}\varphi)^2 - 2(\Pi_{\varepsilon}\varphi)\big(\Pi_{\varepsilon}(X^{\varepsilon}_r(\omega)\varphi)\big)\rangle$$
$$+ \langle \big(\Pi_{\varepsilon}(X^{\varepsilon}_r(\omega)\varphi)\big)^2, 1\rangle - \langle s_{\varepsilon}(\omega)(\Pi_{\varepsilon}X^{\varepsilon}_r(\omega))^2, (\Pi_{\varepsilon}\varphi)^2 - 2(\Pi_{\varepsilon}\varphi)\big(\Pi_{\varepsilon}(X^{\varepsilon}_r(\omega)\varphi)\big)\rangle \, \mathrm{d}r.$$

The proof of the lemma can be found in the appendix.

1.3. **Sparse regime.** The first scaling limit we consider is in a sparse regime, namely when particles of type $\mathfrak{a}$ are rare. For this reason we introduce a sparsity parameter

$$\varrho > \frac{5}{2}d,$$

where the lower bound for the parameter appears for technical reasons (heuristically any positive value of the parameter should work). This parameter should provide the approximation

$$X^{\varepsilon}_t = \varepsilon^{\varrho} Y^{\varepsilon}_t,$$

where the latter process $Y^{\varepsilon}_t$ is of order one. Concretely, the parameter $\varrho$ will appear in the initial condition of the processes and will be appropriately coupled to all other parameters. We thus introduce the following smallness assumption for a sequence $\{X^{\varepsilon,0}\}_{\varepsilon\in(0,1/2)}$ which will later be the sequence of initial conditions for our process.

**Assumption 1.3** (Sparsity)**.** *Fix a $\varrho > \frac{5}{2}d$ and a sequence $X^{\varepsilon,0} \in L^{\infty}(\mathbb{T}^d)$ such that for some $Y^0 \in L^{\infty}(\mathbb{T}^d)$:*

$$0 \leq X^{\varepsilon,0} \leq 1, \qquad \lim_{\varepsilon\to 0} \varepsilon^{-\varrho} X^{\varepsilon,0} = Y^0 \ \textit{in}\ L^{\infty}(\mathbb{T}^d).$$

In addition to imposing sparsity, we consider a special form of selection, such that under appropriate scaling we obtain convergence to space white noise. To obtain a nontrivial scaling in dimension 2 we must keep in mind the renormalization constant $c_{\varepsilon}$ required to solve the parabolic Anderson model:

$$c_{\varepsilon} = \sum_{k\in\mathbb{Z}^2} \frac{\hat{\chi}^2(\varepsilon k)\hat{\chi}_Q(\varepsilon k)}{-\vartheta_{\varepsilon}(k)+1}. \tag{2}$$

The assumptions on the noise are the following.

**Assumption 1.4** (White noise scaling)**.** *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined an i.i.d. sequence of random variables $\{Z_{\varepsilon}(x)\}_{x\in\mathbb{Z}^d_{\varepsilon}}$ with all moments finite and satisfying:*

$$\mathbb{E}\big[Z^2_{\varepsilon}(x)\big] = 1, \qquad Z_{\varepsilon}(\omega, x) \in (-2,2), \qquad \textit{for all}\ \ x\in\mathbb{Z}^d_{\varepsilon},\ \ \varepsilon\in(0,1/2),\ \ \omega\in\Omega.$$

*Then define*

$$s_{\varepsilon}(\omega, y) = Z_{\varepsilon}(\omega, x) - \varepsilon^{\frac{d}{2}} c_{\varepsilon} 1_{\{d=2\}}, \quad \textit{if}\ y\in Q_{\varepsilon}(x), \qquad \forall \omega\in\Omega, x\in\mathbb{T}^d$$

*and write:*

$$\xi^{e}_{\varepsilon}(\omega, x) = \varepsilon^{-\frac{d}{2}} s_{\varepsilon}(\omega, x), \qquad \xi_{\varepsilon}(\omega, x) = \xi^{e}_{\varepsilon}(\omega, x) + c_{\varepsilon}1_{\{d=2\}}.$$

Note how under these assumptions the random field $\xi_{\varepsilon}$ approximates a spatial white noise. Under appropriate scaling, we will prove that the process $X^{\varepsilon}$ converges to a rough superBrownian motion. This process has been introduced and studied by [25] as an approximation of lattice branching process in a static environment. First, recall the construction of the Anderson Hamiltonian, and its relationship to our setting.

**Lemma 1.5.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise $\xi\colon \Omega \to \mathcal{S}'(\mathbb{T}^d)$, that is a process such that for all $f\in\mathcal{S}(\mathbb{T}^d)$ the projection $\langle \xi, f\rangle =: \int_{\mathbb{T}^d} f(x)\xi(\mathrm{d}x)$ are Gaussian random variables with covariance:*

$$\mathbb{E}\Big[\langle \xi, f\rangle\langle \xi, g\rangle\Big] = \langle f, g\rangle, \qquad \forall f, g\in\mathcal{S}(\mathbb{T}^d).$$

*For almost all $\omega \in \Omega$ there exists an operator*

$$\mathcal{H}(\omega) \colon \mathcal{D}_\omega \subseteq C(\mathbb{T}^d) \to C(\mathbb{T}^d),$$

*with a dense domain $\mathcal{D}_\omega \subseteq C(\mathbb{T}^d)$, such that*

$$\mathcal{H}(\omega) = \lim_{\varepsilon \to 0} \left[ \mathcal{A}_\varepsilon + \xi_\varepsilon(\omega) - c_\varepsilon 1_{\{d=2\}} \right] =: \nu_0 \Delta + \xi(\omega) - \infty 1_{\{d=2\}}.$$

*Here the limit should be interpreted in distribution with respect to the probability measure $\mathbb{P}$. The precise sense of the limit is explained in Theorem 1.14. The last notation is just a convenient formalism obtained by exchanging the limit with the sum.*

add proof, especially domain! For now $\mathcal{D}_\omega = \{\text{finite linear combinations of eigenfunctions}\}$

The rough superBrownian motion is then a Markov process conditional on the realization of the spatial white noise and thus of the Anderson Hamiltonian.

**Definition 1.6.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise $\xi$ and $Y^0 \in \mathcal{M}(\mathbb{T}^d)$. A rough superBrownian motion is a couple composed of an enlarged probability space $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^\omega)$ (where $\mathcal{F} \otimes \overline{\mathcal{F}}$ is the product sigma-field and $\mathbb{P}^\omega$ is the conditional law of the process given the realization of the noise, see* Add *for a precise definition of semidirect products) and a map*

$$Y \colon \Omega \times \overline{\Omega} \to C([0, \infty); \mathcal{M}(\mathbb{T}^d)).$$

*Moreover, for $\omega \in \Omega$ let $\{\mathcal{F}^\omega_t\}_{t \geq 0}$ be the filtration generated by $t \mapsto Y_t(\omega)$, right-continuous and enlarged with all null-sets. Then for all $\omega \in \Omega$ such that the operator $\mathcal{H}(\omega)$ is defined (see the lemma above) and for all $\varphi \in \mathcal{D}_\omega$, it is required that the process:*

$$M^\varphi_t := \langle Y_t(\omega), \varphi \rangle - \langle Y^0, \varphi \rangle - \int_0^t \langle Y_s(\omega), \mathcal{H}(\omega)\varphi \rangle \mathrm{d}s$$

*is a centered continuous, square-integrable $\mathcal{F}^\omega_t$-martingale on $[0, T]$ for any $T > 0$ with quadratic variation*

$$\langle M^\varphi \rangle_t = \int_0^t \langle Y_s(\omega), \varphi^2 \rangle \, \mathrm{d}s.$$

In this framework, our main result is the following convergence.

**Theorem 1.7.** *For any $\varrho > \frac{5}{2} d$ consider a random environment $s_\varepsilon$ as in Assumption 1.4, and initial conditions $X^{\varepsilon, 0}$ as in Assumption 1.3. Consider the process $X^\varepsilon$ as in Definition 1.1, associated to the generator:*

$$\varepsilon^{-d-2-\eta} \mathcal{L}(\varepsilon, \varepsilon^{2-\frac{d}{2}} s_\varepsilon(\omega), \varepsilon^\eta),$$

*with $\eta$ defined by:*

$$\eta = \varrho + 2 - d. \tag{3}$$

*Then the process $t \mapsto Y^\varepsilon_t = \varepsilon^{-\varrho} X^\varepsilon_t$ converges in distribution, as a stochastic process on the probability space $(\Omega \times \Omega', \tilde{\mathcal{F}}, \mathbb{P} \ltimes \mathbb{P}^\omega)$ (cf. Definition 1.1):*

$$\lim_{\varepsilon \to 0} Y^\varepsilon = Y \quad \text{in} \quad \mathbb{D}([0, \infty); \mathcal{M}(\mathbb{T}^d)),$$

*where $Y$ is the unique in distribution rough superBrownian motion as in Definition 1.6, started in $Y^0$.*

The coefficients appearing in the rescaling should be interpreted as follows. The term $\varepsilon^{-d-2-\eta}$ guarantees diffusive scaling, namely that a Laplacian appears in the limit. The scaling of $s_\varepsilon$ guarantees convergence to space white noise. Finally the parameter $\eta$ guarantees non-triviality of the limit: it only plays a role in determining the first non-trivial term in the quadratic variation.

1.4. **Diffusive regime.** At this point we consider a different regime. We do not assume any sparsity: instead we perform a diffusive scaling, imposing some specific constraint on the impact parameter $\mathfrak{u}$. This parameter modulates the intensity of the noise driving the system. We assume that $\mathfrak{u} = \varepsilon^{\eta}$ as before. The restriction on $\eta$ is as follows.

**Assumption 1.8.** *Choose $\eta$ such that:*

$$\begin{cases} \eta \geq 1 & \text{if } \ d = 1 \\ \eta > 0 & \text{if } \ d = 2. \end{cases}, \qquad \mu := \eta + d + 2.$$

We still assume that the selection is random, yet we do not consider it space white noise.

**Assumption 1.9.** *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a space white noise $\xi$ on the torus $\mathbb{T}^d$, for $d = 1, 2$. Fix a smooth function $\aleph \in \mathcal{S}(\mathbb{T}^d)$ and define (the latter being only a formal integral):*

$$\bar{\xi}(\omega, x) := \xi(\aleph(x - \cdot)) = \int_{\mathbb{T}^d} \aleph(x - y)\xi(\omega, \mathrm{d}y).$$

*Then write:*

$$s_{\varepsilon}(\omega, x) = \varepsilon^2 \bar{\xi}(\omega, x).$$

Then we define the (stochastic if $d = 1$) FKPP equation in a random potential as follows.

**Definition 1.10.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a white noise $\xi$ and $X^0 \in B^{\alpha}_{2,2}$. A (stochastic if $d = 1$) FKPP process in random potential is a couple given by a probability space $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^{\omega})$ (cf. Definition 1.6) and a map*

$$X \colon \Omega \times \overline{\Omega} \to L^2_{\mathrm{loc}}([0, \infty); B^{\alpha}_{2,2}),$$

*for some $\alpha > 0$. Moreover, for $\omega \in \Omega$ let $\{\mathcal{F}^{\omega}_t\}_{t \geq 0}$ be the filtration generated by $t \mapsto X_t(\omega)$, right-continuous and enlarged with all null-sets. Then for all $\omega \in \Omega$ it is required that, depending on the dimension:*

(1) *In dimension $d = 1$ for all $\varphi \in C^{\infty}(\mathbb{T}^d)$:*

$$N^{\varphi}_t := \langle X_t(\omega), \varphi \rangle - \langle X^0, \varphi \rangle - \int_0^t \langle X_s(\omega), \nu_0 \Delta \varphi \rangle - \langle \bar{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi \rangle \mathrm{d}s$$

*is a continuous in time, square integrable martingale with quadratic variation*

$$\langle N^{\varphi} \rangle_t = \int_0^t \langle X_s(\omega)(1 - X_s(\omega)), \varphi^2 \rangle \mathrm{d}s.$$

(2) *In dimension $d = 2$, $X$ is a solution to:*

$$\begin{aligned} \partial_t X_t(\omega) &= \nu_0 \Delta X_t(\omega) + \bar{\xi}(\omega) X_t(\omega)(1 - X_t(\omega)), \\ X_0(\omega, x) &= X^0(\omega, x), \qquad \forall x \in \mathbb{T}^d. \end{aligned}$$

*The latter should be interpreted in the sense that for all $\varphi \in C^{\infty}(\mathbb{T}^d)$:*

$$\langle X_t(\omega), \varphi \rangle = \langle X^0, \varphi \rangle + \int_0^t \langle X_s(\omega), \nu_0 \Delta \varphi \rangle + \langle \bar{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi \rangle \ \mathrm{d}s.$$

**Remark 1.11.** *Note that in the previous definition, since $X \in L^2_{\mathrm{loc}}([0, \infty); B^{\alpha}_{2,2})$, the quadratic non-linearity:*

$$\int_0^t \langle X^2_s, \varphi \rangle \, \mathrm{d}s$$

*is well-defined. Moreover, up to enlarging the probability space, the process can be represented in $d = 1$ as a solution to an SPDE of the form*

$$\partial_t X = \nu_0 \Delta X + \bar{\xi} X(1 - X) + \sqrt{X(1 - X)} \widetilde{\xi},$$

*where the spatial noise* $\overline{\xi}$ *is independent of the space-time white noise* $\widetilde{\xi}$*, following a classical construction by Konno and Shiga* add *(see also* add *for a similar case in a random environment).*

**Theorem 1.12.** *Let* $\eta, \mu$ *satisfy Assumption 1.8 and* $s_\varepsilon$ *be as in Assumption 1.9. Consider* $X_0 \in \mathcal{S}(\mathbb{T}^d)$*, and let* $X^\varepsilon(\omega)$ *be the Markov process associated to the generator*

$$\varepsilon^{-\mu}\mathcal{L}(\varepsilon, s_\varepsilon(\omega), \varepsilon^\eta)$$

*and started in* $X_0$*, as Definition 1.1. There exists an* $\alpha > 0$ *such that for every* $\omega \in \Omega$

$$\{t \mapsto X_t^\varepsilon(\omega)\}_{\varepsilon \in (0,1/2)}$$

*is tight in the space* $L^2_{\mathrm{loc}}([0,\infty); B^\alpha_{2,2})$*. In particular:*

(1) *In dimension* $d = 1$ *if* $\eta = 1$ *any subsequential limit is a stochastic FKPP process in a random potential as in Definition 1.10.*
(2) *In dimension* $d = 2$ *the entire sequence converges in distribution to an FKPP process in a random potential as in Definition 1.10.*

1.5. **Proof methods.** The proofs of the results we just described are based on a careful study of the operator $\mathcal{A}_\varepsilon$. Intuitively, one expects that this operator approximates the Laplacian with periodic boundary conditions. To quantify this intuition we introduce a division of scales. On large scales, namely for Fourier modes $k$ of order $k \lesssim \frac{1}{\varepsilon}$ we show that $\mathcal{A}_\varepsilon$ has the regularizing properties of the Laplace operator. On small scales, that is for modes of order $k \gtrsim \frac{1}{\varepsilon}$ we do not expect any regularization. Instead we prove that small scales are negligible. To divide small and large scales we use "projection" operators $\mathcal{P}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ on large and small scales respectively. Here we state a slimmed version of the results we require. More technical results are deferred to Section 4 (see in particular ADD for the proof of the following result).

**Theorem 1.13.** *There exists a smooth radial function with compact support* $\daleth \colon \mathbb{R}^d \to \mathbb{R}$ *such that for some* $0 < r < R$*:*

$$\daleth(k) \equiv 1, \ \forall |k| \geq R, \qquad \daleth(k) \equiv 0, \ \forall |k| \leq r,$$

*so that defining*

$$\mathcal{P}_\varepsilon = \daleth(\varepsilon D), \qquad \mathcal{Q}_\varepsilon = (1 - \daleth)(\varepsilon D),$$

*the following hold for any* $\alpha \in \mathbb{R}, p \in [1, \infty]$*:*

- *For any* $\zeta > 0$ *and* $\varphi \in \mathcal{C}^\alpha_p$*:*

$$\mathcal{A}_\varepsilon \varphi \to \nu_0 \Delta \varphi \ \ in \ \ \mathcal{C}_p^{\alpha-2-\zeta}, \qquad as \ \ \varepsilon \to 0,$$

*where*

$$\nu_0 = \frac{1}{12} \quad for \quad d = 1, \qquad \nu_0 = \frac{1}{4\pi} \quad for \quad d = 2. \tag{4}$$

- *Uniformly over* $\lambda > 1, \varepsilon \in (0, 1/2)$ *and* $\varphi \in \mathcal{C}^\alpha_p$ *the following estimates hold:*

$$\|\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1}\varphi\|_{\mathcal{C}_p^{\alpha+2}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1}\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

In particular, a precise control of the regularization effects of the semidiscrete Laplacian $\mathcal{A}_\varepsilon$ allows us to treat semidiscrete approximations of the parabolic Anderson model that appear in the study of the rough superBrownian motion.

**Theorem 1.14.** *Fix* $\kappa > 0$ *and* $\xi_\varepsilon$ *satisfying Assumption 1.4. Up to changing probability space there exists a space white noise* $\xi \colon \Omega \to \mathcal{S}'(\mathbb{T}^d)$ *for which the following hold true for almost all* $\omega \in \Omega$*. The Anderson Hamiltonian*

$$\mathcal{H}(\omega) = \nu_0 \Delta + \xi(\omega) - \infty 1_{\{d=2\}}$$

*associated to $\xi(\omega)$ is defined, as constructed in [2]. The Hamiltonian, as an unbounded self-adjoint operator on $L^2(\mathbb{T}^d)$, has a discrete spectrum given by pairs of eigenvalues and eigenfunctions $\{(\lambda_k(\omega), e_k(\omega))\}_{k\in\mathbb{N}}$ such that:*

$$\lambda_0(\omega) > \lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots, \qquad \lim_{k\to\infty} \lambda_k(\omega) = -\infty, \qquad e_0(\omega, x) > 0, \forall x \in \mathbb{T}^d.$$

*Moreover, for every $k \in \mathbb{N}$, $e_k(\omega) \in \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$, and finite linear combinations of $\{e_k(\omega)\}_{k\in\mathbb{N}}$ form a dense set in $\mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$.*

*For every $k \in \mathbb{N}$ there exists an $\varepsilon_0(\omega, k) \in (0, 1/2)$ such that for every $\varepsilon \leq \varepsilon_0(\omega, k)$ there exists a pair of eigenvalue and associated eigenfunction $(\lambda_k^\varepsilon(\omega), e_k^\varepsilon(\omega))$ for the operator*

$$\mathcal{H}_\varepsilon(\omega) := \mathcal{A}_\varepsilon + (\xi_\varepsilon(\omega) - c_\varepsilon)\Pi_\varepsilon^2, \qquad \mathcal{H}_\varepsilon(\omega) \colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d),$$

*such that*

$$\lim_{\varepsilon\to 0} \lambda_k^\varepsilon(\omega) = \lambda_k(\omega), \qquad \lim_{\varepsilon\to 0} \Pi_\varepsilon e_k^\varepsilon(\omega) = e_k(\omega) \;\; \text{in} \;\; \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d).$$

The proof of this result can be found in Section 5.

### 1.6. Comparison to previous literature.

## 2. Scaling to the rough super-Brownian motion

This section is devoted to the proof of Theorem 1.7. We will leverage the analytic results of Theorem 1.14 to obtain tightness of the sequence $Y^\varepsilon$. Uniqueness of the limit points the follows by a conditional duality argument. Finally, Subsection 2.2 sketches the persistence result for rough superBrownian motion.

### 2.1. Scaling limit.

The core of the tightness proof is condition on the realization of the environment. Since we want to prove convergence in distribution for the sequence $Y^\varepsilon$, the choice of the probability space $\Omega$ of Definition 1.1 is not important. For this reason we adopt the following standing assumption that allows us to work with a suitably chosen probability space.

**Assumption 2.1.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$, the probability space appearing in Definition 1.1 and Assumption 1.4 be such that the results of Theorem 1.14 hold true for almost all $\omega \in \Omega$.*

The first step towards establishing tightness is to restate the martingale problem of Lemma 1.2 to take into account the scaling assumed in Theorem 1.7.

**Lemma 2.2.** *In the setting of Theorem 1.7 and under Assumption 2.1, for every $\omega \in \Omega$ and $\varepsilon \in (0, 1/2)$, under the law $\mathbb{P}^\omega$, and for every $\varphi \in L^1(\mathbb{T}^d)$ the process $t \mapsto \langle Y_t^\varepsilon(\omega), \varphi\rangle$ satisfies the following martingale problem:*

$$\begin{aligned} \langle Y_{t,s}^\varepsilon(\omega), \varphi\rangle =& M_{t,s}^\varepsilon(\varphi) \\ &+ \int_s^t \langle \mathcal{A}_\varepsilon(Y_r^\varepsilon(\omega)) + \Pi_\varepsilon[\xi_\varepsilon(\omega)\Pi_\varepsilon Y_r^\varepsilon(\omega)], \varphi\rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y_r^\varepsilon(\omega)\big)^2, \xi_\varepsilon(\omega)\Pi_\varepsilon(\varphi)\rangle \,\mathrm{d}r, \end{aligned} \tag{5}$$

*where $M_\cdot^\varepsilon(\varphi)$ is a square integrable martingale with predictable quadratic variation given by:*

$$\begin{aligned} \langle M^\varepsilon(\varphi)\rangle_t = \int_0^t &\langle (1+\varepsilon^{2-\frac{d}{2}} s_\varepsilon(\omega))\Pi_\varepsilon Y_r^\varepsilon(\omega), (\Pi_\varepsilon\varphi)^2 - 2\varepsilon^\varrho \Pi_\varepsilon(\varphi)\Pi_\varepsilon(Y_r^\varepsilon(\omega)\varphi)\rangle + \varepsilon^\varrho \langle \big(\Pi_\varepsilon(Y_r^\varepsilon(\omega)\varphi)\big)^2, 1\rangle \\ &- \varepsilon^\varrho \langle \varepsilon^{2-\frac{d}{2}} s_\varepsilon(\omega)(\Pi_\varepsilon Y_r^\varepsilon(\omega))^2, (\Pi_\varepsilon\varphi)^2 - 2\varepsilon^\varrho \Pi_\varepsilon(\varphi)\Pi_\varepsilon(Y_r^\varepsilon(\omega)\varphi)\rangle \,\mathrm{d}r, \end{aligned} \tag{6}$$

**Remark 2.3.** *Note that the only term which is not of lower order in the quadratic variation is*

$$\langle \Pi_\varepsilon Y_r^\varepsilon, \big(\Pi_\varepsilon\varphi\big)^2\rangle,$$

*which, combined with the form of the drift term, provides an algebraic heuristic for obtaining the super-Brownian motion in a static random environment as the scaling limit.*

In order to obtain the convergence, the first step is to prove a tightness result.

**Proposition 2.4.** *In the setting of Theorem 1.7 and under Assumption 2.1 fix any* $\omega \in \Omega$. *For any* $T > 0$ *the sequence* $\{Y^\varepsilon(\omega)\}_{\varepsilon\in(0,1/2)}$ *is tight in* $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$. *Moreover any limit point is continuous, i.e. lies in* $C([0,T];\mathcal{M}(\mathbb{T}^d))$.

The proof will be based on an application of Jakubowski's tightness criterion, which we recall for convenience.

**Proposition 2.5.** *[18, Theorem 3.1] Let* $X$ *be a separable metric space. Let* $F$ *be a family of real, continuous functions on* $X$ *which separates points and is closed under addition. Then a sequence of probability measures* $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ *on* $\mathbb{D}([0,T];X)$ *is tight if the following two conditions are satisfied:*

(1) *For each* $\varepsilon > 0$ *there exists a compact set* $K \subset X$ *such that*
$$\inf_{n\in\mathbb{N}} \mathbb{P}_n\big(X_t \in K,\ \forall t \in [0,T]\big) \geq 1-\varepsilon,$$
*where* $X_t$ *is the canonical process on* $\mathbb{D}([0,T];X)$.

(2) *For each* $f \in F$ *sequence* $\mathbb{P}_n \circ f^{-1}$ *is tight as a measure on* $\mathbb{D}([0,T];\mathbb{R})$.

*Proof of Proposition 2.4.* Since $\omega \in \Omega$ is fixed, we omit writing it, to lighten the notation. The proof is divided into three steps, that allow us to apply a tightness criterion we just recalled. According to this criterion, tightness is given if we prove a compact containment condition, as well as tightness of one-dimensional distributions.

In the first step, we establish the compact containment condition. Since for $R > 0$ sets of the form $K_R = \{\mu\colon \langle \mu, 1\rangle \leq R\} \subseteq \mathcal{M}(\mathbb{T}^d)$ are compact in the weak topology, it is sufficient to show that

$$\forall\delta > 0, \quad \exists R(\delta) > 0,\ \frac{1}{2} > \varepsilon(\delta) > 0 \text{ such that } \inf_{\varepsilon\in(0,\varepsilon(\delta)]} \mathbb{P}\Big(\sup_{t\in[0,T]} \langle Y_t^\varepsilon, 1\rangle \leq R(\delta)\Big) \geq 1-\delta. \tag{7}$$

In the second step, we establish the one-dimensional tightness. Since by Proposition 5.7, it is sufficient to show that for every $k \in \mathbb{N}$ process $\langle Y_t^\varepsilon, e_k\rangle$ is tight in $\mathbb{D}([0,T];\mathbb{R})$. By Aldous' criterion [1, Theorem 1] this reduces to proving that for any sequence of stopping times $\tau_\varepsilon$, taking finitely many values and adapted to the filtration of $Y^\varepsilon$, and any sequence $\delta_\varepsilon$ of constants such that $\delta_\varepsilon \to 0$ for $\varepsilon \to 0$

$$\forall\delta > 0, \qquad \lim_{\varepsilon\to 0} \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, e_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, e_k\rangle| \geq \delta\Big) = 0. \tag{8}$$

In the third step we address the continuity of the limiting process.

*Step 1.* By Proposition 5.7, for any $k \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_0(k)$ there exists an eigenfunction $e_k^\varepsilon$ of $\mathcal{H}_\varepsilon$ such that $\Pi_\varepsilon e_k^\varepsilon \to e_k$ in $\mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$. In particular, since $e_0 > 0$, we may assume that $\Pi_\varepsilon e_0^\varepsilon > 0, \forall \varepsilon \leq \varepsilon_0(0)$ and hence for any positive measure $\mu$ there exists a $C > 0$ such that
$$\langle \mu, 1\rangle \leq C\langle \mu, \Pi_\varepsilon e_0^\varepsilon\rangle, \qquad \forall \varepsilon \leq \varepsilon_0(0).$$
Therefore to show (7) it is sufficient to show that
$$\forall\delta > 0, \quad \exists R(\delta) > 0,\ \varepsilon_0(0) \geq \varepsilon(\delta) > 0 \text{ such that } \inf_{\varepsilon\in(0,\varepsilon(\delta)]} \mathbb{P}\Big(\sup_{t\in[0,T]} \langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle \leq R(\delta)\Big) \geq 1-\delta.$$
We focus our attention on $\langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle$. Through the martingale problem of Lemma 2.2 one obtains
$$\langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle = \langle Y_0^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle + \int_0^t \lambda_0^\varepsilon \langle Y_r^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y_r^\varepsilon\big)^2, \xi_\varepsilon \Pi_\varepsilon^2 e_0^\varepsilon\rangle \,\mathrm{d}r + M_t^\varepsilon(\Pi_\varepsilon e_0^\varepsilon).$$

To treat the nonlinear quadratic term, we shall consider a stopped process. For that purpose fix $R > 0$ and consider a stopping time $\tau_R$ and a parameter $\varrho_0$ defined as
$$\tau_R := \inf\{t \geq 0 \ :\ \langle Y_t^\varepsilon, \Pi_\varepsilon e_0^\varepsilon\rangle \geq R\}, \qquad \varrho_0 = \varrho - \frac{d}{2} - 2d.$$

In view of the stochastic bounds in Lemma 5.6

$$\varepsilon^{\varrho}|\langle(\Pi_\varepsilon Y^\varepsilon_{r\wedge\tau_R})^2, \xi_\varepsilon \Pi^2_\varepsilon e^\varepsilon_0\rangle| \lesssim \varepsilon^{\varrho-\frac{d}{2}-2d}\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle^2 \lesssim R\varepsilon^{\varrho_0}\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle,$$

and therefore

$$\mathbb{E}|\langle Y^\varepsilon_{t\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \lesssim \|Y^\varepsilon_0\|_{L^\infty} + (1+R\varepsilon^{\varrho_0})\int_0^t \mathbb{E}|\langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \,\mathrm{d}r + \mathbb{E}\langle M^\varepsilon(\Pi_\varepsilon e^\varepsilon_0)\rangle_{t\wedge\tau_R}.$$

Furthermore, using the formula for the predictable quadratic variation from Lemma 2.2 one obtains

$$\mathbb{E}\langle M^\varepsilon(\Pi_\varepsilon e^\varepsilon_0)\rangle_{t\wedge\tau_R} \lesssim \mathbb{E}\int_0^t \langle \Pi_\varepsilon Y^\varepsilon_{r\wedge\tau_R}, (\Pi^2_\varepsilon e^\varepsilon_0)^2\rangle + \langle \Pi_\varepsilon(Y^\varepsilon_{r\wedge\tau_R}\Pi_\varepsilon e^\varepsilon_0), \Pi^2_\varepsilon e^\varepsilon_0\rangle \,\mathrm{d}r \lesssim \mathbb{E}\int_0^t \langle Y^\varepsilon_{r\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle \,\mathrm{d}r.$$

Therefore by Gronwall's inequality there exists a $C>0$ such that

$$\sup_{0\le t\le T} \mathbb{E}|\langle Y^\varepsilon_{t\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_0\rangle|^2 \lesssim e^{CR\varepsilon^{\varrho_0}}. \tag{9}$$

It follows that

$$\varepsilon \le R^{\frac{1}{\varrho_0}} \Rightarrow \mathbb{P}\Big(\sup_{0\le t\le T}|\langle Y^\varepsilon_t, \Pi_\varepsilon e^\varepsilon_0\rangle| \ge R\Big) = \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_R\wedge T}, \Pi_\varepsilon e^\varepsilon_0\rangle| = R\Big) \lesssim \frac{1}{R^2}.$$

This concludes the proof of compact containment condition (7).

*Step 2.* Fix $k\in\mathbb{N}$ and $\gamma>0$. In view of calculations from Step 1 there exist $R(\gamma), \varepsilon(\gamma)$ for which (7) holds. Up to choosing a smaller $\varepsilon(\gamma)$ we may also assume that

$$\forall\varepsilon \text{ such that } \varepsilon(\gamma)\ge\varepsilon>0: \qquad \|e_k - \Pi_\varepsilon e^\varepsilon_k\|_{L^\infty} \le \frac{\delta}{2R(\gamma)}.$$

Hence for every $\varepsilon\le\varepsilon(\gamma)$

$$\mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, e_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, e_k\rangle| \ge \delta\Big) \le \gamma + \mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle| \ge \delta\Big).$$

Using representation of Lemma 2.2

$$\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle = \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \lambda^\varepsilon_k\langle Y^\varepsilon_r, \Pi_\varepsilon e^\varepsilon_k\rangle - \varepsilon^{\varrho}\langle\big(\Pi_\varepsilon Y^\varepsilon_r\big)^2, \xi_\varepsilon\Pi^2_\varepsilon e^\varepsilon_k\rangle \,\mathrm{d}r + M^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}(\Pi_\varepsilon e^\varepsilon_k) - M^\varepsilon_{\tau_\varepsilon}(\Pi_\varepsilon e^\varepsilon_k).$$

Hence one obtains (writing for simplicity $R$ instead of $R(\gamma)$):

$$\mathbb{P}\Big(|\langle Y^\varepsilon_{\tau_\varepsilon+\delta_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon}, \Pi_\varepsilon e^\varepsilon_k\rangle| \ge \delta\Big) \le \gamma + \mathbb{P}\Big(|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle| \ge \delta\Big).$$

Computations analoguous to those in Step 1. guarantee that

$$\mathbb{P}\Big(|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle| \ge \delta\Big) \le \frac{1}{\delta^2}\mathbb{E}\Big[\big|\langle Y^\varepsilon_{(\tau_\varepsilon+\delta_\varepsilon)\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle - \langle Y^\varepsilon_{\tau_\varepsilon\wedge\tau_R}, \Pi_\varepsilon e^\varepsilon_k\rangle\big|^2\Big] \lesssim \delta_\varepsilon.$$

Since $\gamma$ is arbitrary, this proves (8).

*Step 3.* So far any limit point $Y$ of the sequence $Y^\varepsilon$ lies in the Skorohod space $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$. Since $\mathcal{M}(\mathbb{T}^d)$ is endowed with the weak topology, to prove that actually $Y\in C([0,T];\mathcal{M}(\mathbb{T}^d))$, it is sufficient to show that for any continuous function $\varphi$, $\langle Y_t, \varphi\rangle$ is continuous in time. Here one can apply a criterion [14, Theorem 3.10.2] according to which it is sufficient to prove that the maximum size of a jump converges weakly to zero. In our case such convergence is even almost sure, since:

$$|\langle Y^\varepsilon_t, \varphi\rangle - \langle Y^\varepsilon_{t-}, \varphi\rangle| \lesssim \varepsilon^d\|\varphi\|_{C(\mathbb{T}^d)}.$$

This follows from the definition of the generator, as well as the exact definition of $\eta$ (cf. Equation (3)) jumps are bounded as follows:

$$\|Y^\varepsilon_t - Y^\varepsilon_{t-}\|_{L^\infty} \lesssim \varepsilon^{2-d} \lesssim 1.$$

Since a jump has an impact only in a ball $B_\varepsilon(x)$ for some $x \in \mathbb{T}^d$, integrating $\varphi$ over such ball guarantees the previous bound. □

Finally we are in position to deduce Theorem 1.7.

*Proof of Theorem 1.7.* By Proposition 2.4 the sequence $Y_\varepsilon(\omega)$ is tight, for every $\omega \in \Omega$, under Assumption 2.1 (recall that we can always put ourselves in the setting of this assumption by changing probability space, which does not affect the convergence in distribution). It remains to show that, for fixed realization $\omega \in \Omega$, every limit point satisfies the martingale problem for the rough superBrownian motion as in Definition 1.6 (Step 1 below), and that solutions to such martingale problems are unique (Step 2 below).

*Step 1.* As in the previous proof, since $\omega \in \Omega$ is fixed we omit writing it. Moreover it is sufficient to fix a finite but arbitrary time horizon $T > 0$ and check the martingale property until that time. Assume that (up to taking a subsequence and applying the Skorohod representation theorem) $Y^\varepsilon \to Y$ almost surely in $\mathbb{D}([0,T];\mathcal{M}(\mathbb{T}^d))$. Since $\mathcal{D}_\omega$ is composed of finite linear combinations of eigenfunctions, it is sufficient to prove the martingale property of Definition 1.6 for $\varphi = e_k$ for some $k \in \mathbb{N}$. In this setting, one has that almost surely:

$$\begin{aligned}
M^{e_k}_t &= \langle Y_{t,0}, e_k\rangle - \int_0^t \langle Y_s, \mathcal{H}e_k\rangle \,\mathrm{d}s = \langle Y_{t,0}, e_k\rangle - \lambda_k \int_0^t \langle Y_s, e_k\rangle \,\mathrm{d}s \\
&= \lim_{\varepsilon\to 0}\left[\langle Y^\varepsilon_{t,0}, \Pi_\varepsilon e^\varepsilon_k\rangle - \int_0^t \langle \mathcal{A}_\varepsilon(Y^\varepsilon_r) + \Pi_\varepsilon[\xi_\varepsilon \Pi_\varepsilon Y^\varepsilon_r], \Pi_\varepsilon e^\varepsilon_k\rangle - \varepsilon^\varrho \langle \big(\Pi_\varepsilon Y^\varepsilon_r\big)^2, \xi_\varepsilon \Pi^2_\varepsilon e^\varepsilon_k\rangle \,\mathrm{d}r\right] \\
&= \lim_{\varepsilon\to 0} M^\varepsilon_t(\Pi_\varepsilon e^\varepsilon_k).
\end{aligned}$$

The convergence of the linear terms in the second line is a consequence of the convergence

$$\Pi_\varepsilon e^\varepsilon_k \to e_k \ \text{ in } \ \mathcal{C}^{2-\frac{d}{2}-\kappa}, \qquad \lambda^\varepsilon_k \to \lambda_k$$

as proved in Proposition ?? (where also the eigenpairs $e^\varepsilon_k, \lambda^\varepsilon_k$ are defined). As for the non-linear term, one has as in the previous proof:

$$\langle \big(\Pi_\varepsilon Y^\varepsilon_r\big)^2, \xi_\varepsilon \Pi^2_\varepsilon e^\varepsilon_k\rangle \lesssim \varepsilon^{\varrho - 2d - \frac{d}{2}} \langle Y^\varepsilon_r, 1\rangle^2 \to 0,$$

by the assumption on $\varrho$. To prove that $M^{e_k}$ is a martingale, one has to show that such property is conserved when passing to the limit. For $R > 0$ consider the stopping times

$$\begin{aligned}
\tau_R(Y) &= \inf\{t \geq 0 \mid \langle Y_t, 1\rangle \geq R\} \\
&= \lim_{\varepsilon\to 0} \inf\{t \geq 0 \mid \langle Y^\varepsilon_t, 1\rangle \geq R\} =: \lim_{\varepsilon\to 0} \tau_R(Y^\varepsilon).
\end{aligned}$$

This sequence is localizing, in the sense that it makes $M^{e_k}$ a local martingale with quadratic variation:

$$\begin{aligned}
\langle M^{e_k}_{\cdot\wedge\tau_R(Y)}\rangle_t = \lim_{\varepsilon\to 0}\Bigg[ \int_0^{t\wedge\tau_R(Y^\varepsilon)} & \langle (1+\varepsilon^{2-\frac{d}{2}} s_\varepsilon)\Pi_\varepsilon Y^\varepsilon_r, (\Pi^2_\varepsilon e^\varepsilon_k)^2 - 2\varepsilon^\varrho \Pi^\varepsilon_\varepsilon(e^\varepsilon_k)\Pi_\varepsilon(Y^\varepsilon_r \Pi_\varepsilon e^\varepsilon_k)\rangle \\
&+ \varepsilon^\varrho \langle \big(\Pi_\varepsilon(Y^\varepsilon_r \Pi_\varepsilon e^\varepsilon_k)\big)^2, 1\rangle \\
&- \varepsilon^\varrho \langle \varepsilon^{2-\frac{d}{2}} s_\varepsilon (\Pi_\varepsilon Y^\varepsilon_r)^2, (\Pi^2_\varepsilon e^\varepsilon_k)^2 - 2\varepsilon^\varrho \Pi^2_\varepsilon(e^\varepsilon_k)\Pi_\varepsilon(Y^\varepsilon_r \Pi_\varepsilon e^\varepsilon_k)\rangle \,\mathrm{d}r \Bigg] \\
= \int_0^{t\wedge\tau_R} \langle Y_s, e^2_k\rangle \,\mathrm{d}s. &
\end{aligned}$$

To conclude that $M^{e_k}$ is itself a square integrable martingale it suffices to observe that:

$$\sup_{0\leq t\leq T} \mathbb{E}|\langle Y_t, 1\rangle|^2 < \infty,$$

which follows by applying Fatou's lemma, first over $\varepsilon$ and then over $R$, to Equation (9) in the previous proof.

*Step 2.* We conclude by explaining the uniqueness in law of a process $Y$ satisfying the martingale problem of the rough superBrownian motion (in the following as always $\omega \in \Omega$ is fixed, and we omit from writing it. In particular, all averages are still conditional on the realization of the environment). The uniqueness is the consequence of a duality argument. For any $\varphi \geq 0, \varphi \in C^\infty$ we will find a process $t \mapsto U_t\varphi$ such that:

$$\mathbb{E}\Big[e^{-\langle Y_t, \varphi\rangle}\Big] = e^{-\langle Y^0, U_t\varphi\rangle}. \tag{10}$$

Hence the distribution of $\langle Y_t, \varphi\rangle$ is uniquely characterized by its Laplace transform. This also characterizes the law of the entire process $\langle Y_t, \varphi\rangle$ through a Dynkin-type argument (see [12, Lemma 3.2.5]), proving the required result.

We are left with the task of describing the process $U_t\varphi$. This is the solution, evaluated at time $t \geq 0$, of the following nonlinearly damped parabolic equation:

$$\partial_t(U.\varphi) = \mathcal{H}(U.\varphi) - \frac{1}{2}(U.\varphi)^2, \qquad U_0\varphi = \varphi,$$

where we consider solutions in the mild sense, namely:

$$U_t\varphi = e^{t\mathcal{H}}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathcal{H}}(U_s\varphi)^2 \,\mathrm{d}s,$$

as constructed in Lemma 5.10. To obtain Equation (10) consider some $\zeta > 0$ and a process $\psi \in C([0,T]; \mathcal{C}^\zeta)$ of the form:

$$\psi_t = e^{t\mathcal{H}}\psi_0 + \int_0^t e^{(t-s)\mathcal{H}} f_s \,\mathrm{d}s,$$

with $f \in C([0,T]; \mathcal{C}^\zeta), \psi_0 \in \mathcal{C}^\zeta$. Approximating $f$ through a piece-wise constant function in time $\widetilde{f}$ and approximating both $\widetilde{f}$ and $\varphi$ via a finite number of eigenvalues in view of Lemma 5.9, and using the continuity of the semigroup as in Equation (45), it follows from the definition of the rough superBrownian motion that for $0 \leq s \leq t$:

$$\langle Y_s, \psi_{t-s}\rangle - \langle Y_0, \psi_t\rangle - \int_0^s \langle Y_r, f_r\rangle \,\mathrm{d}r =: \widetilde{M}_s(\psi)$$

is a continuous martingale with quadratic variation:

$$\langle \widetilde{M}(\psi)\rangle_s = \int_0^s \langle Y_r, \psi_{t-r}^2\rangle \,\mathrm{d}r.$$

Now we apply this observation together with Itô's formula to deduce that

$$[0,t] \ni s \mapsto e^{-\langle Y_s, U_{t-s}\varphi\rangle}$$

is a martingale on $[0,t]$. In particular, this implies Equation (10) and conclude the proof. □

2.2. **Persistence.** In this section we shortly discuss the persistence of the rough superBrownian motion on a torus. The discussion is very similiar to [25], but we sketch it here as this result is crucial for our biological motivation.

## 3. Scaling to FKPP

As in the previous case, throughout this section we will fix one realization $\omega$ of the environment and work conditional on that realization. Unlike in the previous section we do not need to change probability space.

The first step towards the scaling limit is to restate the martingale problem of Lemma 1.2 in the current setting (the proof is an immediate consequence of the mentioned lemma).

**Lemma 3.1.** *Under the assumptions of Theorem 1.12 fix any $\omega \in \Omega$. For all $\varphi \in L^1(\mathbb{T}^d)$, the process $t \mapsto \langle X^\varepsilon_t, \varphi \rangle$ satisfies that:*

$$\langle X^\varepsilon_{t,s}(\omega), \varphi \rangle = \int_s^t \langle \mathcal{A}_\varepsilon(X^\varepsilon_r(\omega)), \varphi \rangle + \langle \Pi_\varepsilon \big[\bar{\xi}(\omega)(\Pi_\varepsilon X^\varepsilon_r(\omega) - (\Pi_\varepsilon X^\varepsilon_r(\omega))^2)\big], \varphi \rangle \, \mathrm{d}r + M^\varepsilon_{t,s}(\varphi), \tag{11}$$

*where $M^\varepsilon_\cdot(\varphi)$ is a centered square integrable martingale with predictable quadratic variation:*

$$\begin{aligned}\langle M^\varepsilon(\varphi)\rangle_t = \varepsilon^{\eta+d-2} \int_0^t & \langle (1+s_\varepsilon(\omega))\Pi_\varepsilon X^\varepsilon_r(\omega), (\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\rangle \\ & + \langle \big(\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\big)^2, 1\rangle - \langle s_\varepsilon(\omega)(\Pi_\varepsilon X^\varepsilon_r(\omega))^2, (\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r(\omega)\varphi)\rangle \, \mathrm{d}r.\end{aligned} \tag{12}$$

Now we are able to show tightness for the process.

**Proposition 3.2.** *Under the assumptions of Theorem 1.12 fix any $\omega \in \Omega$. Fix moreover any $T > 0$ and $\alpha$ such that:*

$$\begin{cases} \alpha \in (0, 1/2) & \text{if } d = 1, \\ \alpha \in (0, \eta) & \text{if } d = 2. \end{cases}$$

*The sequence $\{s \mapsto \Pi_\varepsilon X^\varepsilon_s(\omega)\}_{\varepsilon \in (0,1/2)}$ is tight in the space:*

$$L^2([0,T]; B^\alpha_{2,2}).$$

*In addition, the sequence $\{s \mapsto X^\varepsilon_s(\omega)\}_{\varepsilon \in (1,1/2)}$ is tight in $\mathbb{D}([0,T]; \mathcal{M}(\mathbb{T}^d))$, and any limit point lies in $C([0,T], \mathcal{M}(\mathbb{T}^d))$.*

A crucial step in the proof of Proposition 3.2 is a compactness criterion due to Simon, which we recall for convenience. Here the space $W^{2,\zeta}([0,T];Y) \subset L^2([0,T];Y)$ is defined by the Sobolev-Slobodeckij norm

$$\|f\|_{W^{2,\zeta}([0,T];Y)} = \|f\|_{L^2([0,T];Y)} + \left( \int_0^T \int_0^T \frac{\|f(t) - f(r)\|_Y^2}{|t-r|^{2\zeta+1}} \, \mathrm{d}t \, \mathrm{d}r \right)^{\frac{1}{2}}.$$

**Proposition 3.3** (Corollary 5, [28])**.** *Let $X, Y, Z$ be three Banach spaces such that $X \subset Y \subset Z$ with the embedding $X \subset Y$ being compact. Then also the following embeddinf is compact, for any $s > 0$:*

$$L^p([0,T];X) \cap W^{s,p}([0,T];Z) \subseteq L^p([0,T];Y).$$

Now, we pass to the proof of tightness.

*Proof of Proposition 3.2.* Since $\omega \in \Omega$ is fixed throughout the proof, we omit writing it, to lighten the notation. Tightness of the sequence $X^\varepsilon$ in $\mathbb{D}([0,T]; \mathcal{M}(\mathbb{T}^d))$ is an immediate consequence of the bound $0 \le X^\varepsilon_t \le 1$. To show that moreover any limit point lies in $C([0,T]; \mathcal{M}(\mathbb{T}^d))$ notice that for any $\varphi \in C(\mathbb{T}^d)$

$$|\langle X^\varepsilon_t, \varphi \rangle - \langle X^\varepsilon_{t-}, \varphi \rangle| \lesssim \varepsilon^{\eta+d} \|\varphi\|_{L^\infty},$$

so that the maximal jump size is vanishing as $\varepsilon \to 0$. The continuity of the limit points follows then through [14, Theorem 3.10.2].

Therefore we now concentrate on proving the tightness of the sequence $\Pi_\varepsilon X^\varepsilon_s$. For simplicity, let us define the parameter $\lambda$ as follows:

$$\begin{cases} \text{If } d=1, & \eta = 1 \quad \Rightarrow \quad \text{define } \lambda = 0, \\ \text{If } d=2, & \eta = 0 \quad \Rightarrow \quad \text{define } \lambda = \eta. \end{cases} \tag{13}$$

Our aim is to apply Proposition 3.3 with:

$$X = B^{\alpha'}_{2,2}, \quad Y = B^{\alpha}_{2,2}, \quad Z = B^{\alpha''}_{2,2},$$

for appropriate $\alpha' > \alpha > \alpha''$.

*Step 1.* First, we derive a uniform bound for the second moment of the $B^\alpha_{2,2}$ norm (this in particular implies boundedness of the sequence $\Pi_\varepsilon X^\varepsilon$ in $L^2([0,T]; B^\alpha_{2,2})$):

$$\sup_{\varepsilon \in (0,1/2)} \sup_{0 \le t \le T} \mathbb{E} \|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}} < \infty. \tag{14}$$

To obtain this bound it is convenient to prove the following (stronger) bound, uniformly over $s \in [0,T]$

$$\sup_{s \le t \le T} \mathbb{E}\big[\|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}} \big| \mathcal{F}_s\big] \lesssim_T 1 + \|\Pi_\varepsilon X^\varepsilon_s\|^2_{B^\alpha_{2,2}}, \tag{15}$$

where $\{\mathcal{F}_t\}_{t\ge 0}$ is the natural filtration generated by $X^\varepsilon$ (we omit the dependence on $\varepsilon$). We state the bound with the conditional expectation, since in this form it is simpler to derive, via a Gronwall-type argument. For brevity, fix the notation

$$\overline{X}^\varepsilon = \Pi_\varepsilon X^\varepsilon.$$

By the martingale representation of Lemma 3.1 and a change of variables formula

$$\overline{X}^\varepsilon_t = e^{(t-s)\mathcal{A}_\varepsilon} \overline{X}^\varepsilon_s + \int_s^t e^{(t-r)\mathcal{A}_\varepsilon} \Pi^2_\varepsilon \big[\xi\big(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2\big)\big] \, \mathrm{d}r + \int_{s+}^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon} \, \mathrm{d}M^\varepsilon_r,$$

where the last integral is understood as a martingale measure (cf. [33]). However, for the purpose of the proof it is sufficient to consider its one dimensional projections, that is for $\varphi \in C(\mathbb{T}^d)$

$$\langle \overline{X}^\varepsilon_t, \varphi\rangle = \langle \overline{X}^\varepsilon_s, e^{(t-s)\mathcal{A}_\varepsilon}\varphi\rangle + \int_s^t \langle \Pi^2_\varepsilon \big[\xi\big(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2\big)\big], e^{(t-r)\mathcal{A}_\varepsilon}\varphi\rangle \, \mathrm{d}r + \int_{s+}^t \mathrm{d}M^\varepsilon_r(\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\varphi).$$

The $B^\alpha_{2,2}$ norm is estimated by

$$\begin{aligned} \mathbb{E}\big[\|\overline{X}^\varepsilon_t\|^2_{B^\alpha_{2,2}} \big| \mathcal{F}_s\big] \lesssim \|\overline{X}^\varepsilon_s\|^2_{B^\alpha_{2,2}} &+ \mathbb{E}\bigg[\bigg\|\int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\Pi^2_\varepsilon\big[\xi(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2)\big] \, \mathrm{d}r\bigg\|^2_{B^\alpha_{2,2}} \bigg| \mathcal{F}_s\bigg] \\ &+ \mathbb{E}\bigg[\bigg\|\int_{s+}^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon} \, \mathrm{d}M^\varepsilon_r\bigg\|^2_{B^\alpha_{2,2}} \bigg| \mathcal{F}_s\bigg]. \end{aligned}$$

An extension of the paraproduct estimates of Lemma 5.1 to the $B^\alpha_{p,q}$ scale (see [3, Theorems 2.82, 2.85]) guarantees that

$$\|f^2\|_{B^\alpha_{2,2}} \le 2\|f \varolessthan f\|_{B^\alpha_{2,2}} + \|f \odot f\|_{B^\alpha_{2,2}} \lesssim \|f\|_{L^\infty}\|f\|_{B^\alpha_{2,2}}$$

and through the Schauder estimates of Proposition 4.16, the $L^\infty$ bound on $\overline{X}^\varepsilon$ and the fact that $\bar{\xi}$ is smooth one obtains

$$\mathbb{E}\bigg[\bigg\|\int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\Pi^2_\varepsilon\big[\bar{\xi}(\overline{X}^\varepsilon_r - (\overline{X}^\varepsilon_r)^2)\big] \, \mathrm{d}r\bigg\|^2_{B^\alpha_{2,2}} \bigg| \mathcal{F}_s\bigg] \lesssim |t-s| \sup_{s \le t \le T} \mathbb{E}\big[\|\overline{X}^\varepsilon_t\|^2_{B^\alpha_{2,2}} \big| \mathcal{F}_s\big].$$

As for the martingale term, by the definition of the space $B^\alpha_{2,2}$ one has:

$$\varepsilon^{2\lambda}\mathbb{E}\bigg[\bigg\|\int_{s+}^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon} \, \mathrm{d}M^\varepsilon_r\bigg\|^2_{B^\alpha_{2,2}} \bigg| \mathcal{F}_s\bigg] = \sum_{j \ge -1} 2^{2\alpha} \int_{\mathbb{T}^d} \varepsilon^{2\lambda}\mathbb{E}\bigg[\bigg|\int_{s+}^t \mathrm{d}M^\varepsilon_r(e^{(t-r)\mathcal{A}_\varepsilon}\Pi_\varepsilon K^x_j)\bigg|^2 \ \bigg| \ \mathcal{F}_s\bigg] \mathrm{d}x.$$

Using the predictable quadratic variation computed in Lemma 3.1 one obtains, uniformly over $x$

$$
\begin{aligned}
(16)\quad & \varepsilon^{2\lambda}\mathbb{E}\Big[\Big|\int_{s+}^{t} \mathrm{d}M^{\varepsilon}_r(e^{(t-r)\mathcal{A}_\varepsilon}\Pi_\varepsilon K^x_j)\Big|^2 \;\Big|\; \mathcal{F}_s\Big] \\
&= \varepsilon^{2\lambda}\mathbb{E}\Big[\int_s^t \langle \overline{X}^{\varepsilon}_r, (1+s_\varepsilon)\big[(\Pi^2_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)^2 - 2\Pi^2_\varepsilon(e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)\Pi_\varepsilon(X^\varepsilon_r\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)\big]\rangle \\
&\qquad + \langle \big(\Pi_\varepsilon(X^\varepsilon_r\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)\big)^2, 1\rangle \\
&\qquad - \langle (\overline{X}^\varepsilon_r)^2, s_\varepsilon\big[(\Pi^2_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)^2 - 2\Pi^2_\varepsilon(e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)\Pi_\varepsilon(X^\varepsilon_r\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j)\big]\rangle \,\mathrm{d}r \;\Big|\; \mathcal{F}_s\Big] \\
&\lesssim \varepsilon^{2\lambda}\int_s^t \|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j|\|^2_{L^2}\,\mathrm{d}r,
\end{aligned}
$$

since $|s_\varepsilon|, |X^\varepsilon| \le 1$. Now, for $\zeta \in \mathbb{R}$, for example via the Poisson summation formula in Lemma 4.1 and a scaling argument on $\mathbb{R}^d$:

$$\|K^x_j\|_{\mathcal{C}^\zeta_1} \lesssim 2^{j\zeta}$$

and therefore by the Schauder estimates of Proposition 4.16 and Lemma 4.10, for $\gamma \in (0,1)$

$$\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j\|_{\mathcal{C}^{\zeta+\gamma}_1} \lesssim (t-r)^{-\frac{\gamma}{2}}2^{\zeta}.$$

For clarity, dimension $d=1$ and dimension $d=2$ are treated separately. In dimension $d=1$ choose $-\frac{1}{2} < \zeta < -\alpha$ and fix $\gamma \in (0,1)$ such that $\zeta + \gamma > \frac{1}{2}$. Then, by Besov embeddings, one has

$$\|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j|\|^2_{L^2} \le \|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j\|^2_{L^2} \lesssim \|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j\|^2_{\mathcal{C}^{\zeta+\gamma}_1} \lesssim (t-r)^{-\gamma}2^{2\zeta}.$$

In dimension $d=2$, where $\eta = \lambda$, choose $\kappa > 0$ such that $\alpha < \eta - 5\kappa$ and set $\gamma = 1-\kappa$. Then Lemma 4.10 and Besov embeddings 4.2 guarantee that

$$
\begin{aligned}
\|\Pi_\varepsilon|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j|\|_{L^2} &\lesssim \varepsilon^{\eta-\kappa}\||\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j|\|_{\mathcal{C}^{-\eta+2\kappa}_2} \lesssim \varepsilon^{\eta-\kappa}\||\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j|\|_{\mathcal{C}^{-\kappa}_{\frac{2}{1+\eta-3\kappa}}} \\
&\lesssim \varepsilon^{\eta-\kappa}\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j\|_{L^{\frac{2}{1+\eta-3\kappa}}} \lesssim \varepsilon^{\eta-\kappa}\|\Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}K^x_j\|_{\mathcal{C}^{1-\eta+4\kappa}_1} \\
&\lesssim \varepsilon^{\eta-\kappa}(t-r)^{\frac{1-\kappa}{2}}\|K^x_j\|_{\mathcal{C}^{-\eta+5\kappa}_1} \lesssim \varepsilon^{\eta-\kappa}(t-r)^{\frac{\gamma}{2}}2^{-j(\eta-5\kappa)}.
\end{aligned}
$$

In both dimensions, substituting the estimate into (16) one obtains

$$\mathbb{E}\Big[\Big\|\int_s^t \Pi_\varepsilon e^{(t-r)\mathcal{A}_\varepsilon}\,\mathrm{d}M^\varepsilon_r\Big\|^2_{B^\alpha_{2,2}}\Big|\mathcal{F}_s\Big] \lesssim |t-s|^{1-\gamma}.$$

For sufficiently small, deterministic $T^*$, chosen uniform over all parameters, inequality (15) is shown for all $(t-s) \le T^*$. Due to the presence of the conditional expectation, one can exploit this argument for general $t, s$ via a Gronwall-type argument. Indeed, to extend the estimate to $2T^*$, observe there exists a $C(T^*)$ such that

$$
\begin{aligned}
\sup_{t\in[s,s+2T^*]}\mathbb{E}\big[\|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}}\big|\mathcal{F}_s\big] &\le C(T^*)\Big(1+\sup_{t\in[s,s+T^*]}\mathbb{E}\big[\|\Pi_\varepsilon X^\varepsilon_t\|^2_{B^\alpha_{2,2}}\big|\mathcal{F}_s\big]\Big) \\
&\le C(T^*)\Big(1+C(T^*)\Big(1+\mathbb{E}\big[\|\Pi_\varepsilon X^\varepsilon_s\|^2_{B^\alpha_{2,2}}\big]\Big)\Big).
\end{aligned}
$$

Iterating this argument yields the bound for arbitrary $T$.

*Step 2.* The next goal is a bound for the expectation of an increment. For this reason fix

$$0 < \beta < \alpha,$$

with $\alpha$ as in Step 1. We shall prove that there exists a $\zeta > 0$ satisfying:

$$\mathbb{E}\Big[\|\overline{X}^\varepsilon_t - \overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}\Big] \lesssim |t-s|^{4\zeta}. \tag{17}$$

Indeed, arguments similiar to those in Step 1. show that

$$\begin{aligned}\mathbb{E}\Big[\|\overline{X}^\varepsilon_t - \overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}\Big] &\leq \mathbb{E}\Big[\|\overline{X}^\varepsilon_t - e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}\Big] + \mathbb{E}\Big[\|e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}^\varepsilon_s - \overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}\Big] \\ &\lesssim \mathbb{E}\Big[\|\overline{X}^\varepsilon_t - e^{(t-s)\mathcal{A}_\varepsilon}\overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}\Big] + |t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}^\varepsilon_s\|^2_{B^\alpha_{2,2}} \\ &\lesssim |t-s|^{1-\gamma}(1+\mathbb{E}\|\overline{X}^\varepsilon_s\|^2_{B^\beta_{2,2}}) + |t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}^\varepsilon_s\|^2_{B^\alpha_{2,2}},\end{aligned}$$

where the penultimate step follows from Lemma 4.17. This is enough to establish (17).

*Step 3.* Notice that (14) and (17) together guarantee that

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\Big[\|\overline{X}^\varepsilon\|^2_{L^2([0,T];B^\alpha_{2,2})} + \|\overline{X}^\varepsilon\|_{W^{2,\zeta}([0,T];B^\beta_{2,2})}\Big] < \infty,$$

with $\zeta$ as in (17). Note that this implies tightness in $L^2([0,T];B^{\alpha'}_{2,2})$ for any $\alpha' < \alpha$, which is still sufficient for the result, since $\alpha$ varies in an open set.

□

At this point, the last step is to prove that any limit point satisfies the required martingale problem (in $d=1$) or solves the required PDE (in $d=2$).

*Proof of Theorem Theorem 1.12.* As in all previous cases, we fix $\omega\in\Omega$ and do not state explicitly the dependence on it. We treat the drift and the martingale part differently.

*Step 1.* We start with the drift, which is the same in both dimensions. Since Let $X$ be any limit point of $X^\varepsilon$ in $C([0,T];\mathcal{M}(\mathbb{T}^d))$. The previous proposition guarantees that any such $X$ lies almost surely in $L^2([0,T];B^\alpha_{2,2})$ for some $\alpha>0$. In addition, through Sorohod representation, we can assume that $\Pi_\varepsilon X^\varepsilon \to X$ in $L^2([0,T];B^\alpha_{2,2})$ almost surely. In particular, for $\varphi\in C^\infty(\mathbb{T}^d)$, defining:

$$N^\varphi_t = \langle X_{t,0},\varphi\rangle - \int_0^t \langle X_s, \nu_0\Delta\varphi\rangle + \langle\bar{\xi}(X_s - X_s^2),\varphi\rangle\,\mathrm{d}s,$$

and since regarding the nonlinear term one can estimate:

$$\int_0^t\int_{\mathbb{T}^d}|X_s^2 - (\Pi_\varepsilon X^\varepsilon)^2|\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t\int_{\mathbb{T}^d} 2|X_s - \Pi_\varepsilon X^\varepsilon|\,\mathrm{d}x\,\mathrm{d}s \lesssim \|X_s - \Pi_\varepsilon X^\varepsilon\|_{L^2([0,T];B^\alpha_{2,2})}$$

and applying Lemma 4.14, one has almost surely:

$$\begin{aligned}N^\varphi_t &= \lim_{\varepsilon\to0}\Big[\langle\Pi_\varepsilon X^\varepsilon_{t,0},\varphi\rangle - \int_0^t\langle\mathcal{A}_\varepsilon X^\varepsilon_s,\varphi\rangle + \langle\overline{\xi}\big[\Pi_\varepsilon X^\varepsilon_s - (\Pi_\varepsilon X^\varepsilon_s)^2\big],\Pi^2_\varepsilon\varphi\rangle\,\mathrm{d}s\Big] \\ &=: \lim_{\varepsilon\to0} N^{\varphi,\varepsilon}_t.\end{aligned}$$

*Step 2.* Now we prove that $N^\varphi_t$ is a martingale, with quadratic variation depending on the dimension. In $d=2$ the quadratic variation will be zero and hence $N^\varphi\equiv0$, proving that the limit is deterministic (conditional on the environment). Since $N^{\varepsilon,\varphi}_t$ is a sequence of martingales, by Lemma 3.1, the fact that also $N^\varphi_t$ is a martingale follows from the uniform bound of Equation (14). The quadratic variation of $N^{\varepsilon,\varphi}$ is given by:

$$\begin{aligned}\langle N^{\varepsilon,\varphi}\rangle_t = \varepsilon^\lambda\int_0^t &\langle(1+s_\varepsilon)\Pi_\varepsilon X^\varepsilon_r, (\Pi^2_\varepsilon\varphi)^2 - 2\Pi^2_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r\varphi)\rangle \\ &+ \langle\big(\Pi_\varepsilon(X^\varepsilon_r\Pi_\varepsilon\varphi)\big)^2,1\rangle - \langle s_\varepsilon(\Pi_\varepsilon X^\varepsilon_r)^2,(\Pi^2_\varepsilon\varphi)^2 - 2\Pi^2_\varepsilon(\varphi)\Pi_\varepsilon(X^\varepsilon_r\Pi_\varepsilon\varphi)\rangle\,\mathrm{d}s,\end{aligned}$$

with $\lambda$ as in Equation (13). Passing to the limit one has:

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle N^{\varepsilon,\varphi}\rangle_t &= 1_{\{\lambda=0\}}\int_0^t \langle X_s, \varphi^2 - 2X_s\varphi^2\rangle + \langle X_s^2, \varphi^2\rangle \,\mathrm{d}s \\ &= 1_{\{\lambda=0\}}\int_0^t \langle X_s(1-X_s), \varphi^2\rangle \,\mathrm{d}s.\end{aligned}$$

This is of the required form for Theorem 1.12. Moreover, a localization argument guarantees that that $\langle N^\varphi\rangle_t = \lim_{\varepsilon\to 0}\langle N^{\varepsilon,\varphi}\rangle_t$, thus completing the proof.

□

## 4. Schauder Estimates

In this section we will introduce the some relevant notions regarding functions spaces. Then the discussion will focus on the analysis of Sobolev regularity of characteristic functions and commutator estimates. Our aim is to provide an accessible and, as far as possible, complete exposition of the results.

### 4.1. **Functional analysis & charactersic functions.**

In this subsection we state some results on Fourier Analysis and Besov spaces which are used throughout the paper.

We begin by stating Poisson summation formula. The proof is omitted, it can be found, for example, in [20], Section 6.1.15.

**Lemma 4.1.** *For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ it holds that:*

$$\mathcal{F}^{-1}_{\mathbb{T}^d}\varphi(x) = \sum_{z\in\mathbb{Z}^d} \mathcal{F}^{-1}_{\mathbb{R}^d}\varphi(x+z).$$

*In particular, this implies for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the bound:*

$$\|\mathcal{F}^{-1}_{\mathbb{T}^d}\varphi\|_{L^1(\mathbb{T}^d)} \le \|\mathcal{F}^{-1}_{\mathbb{R}^d}\varphi\|_{L^1(\mathbb{R}^d)}.$$

Recall that the Besov spaces $B^\alpha_{p,q}(\mathbb{T}^d)$ are defined via a dyadic partition of the unity $\{\varrho_j\}_{j\ge -1}$ such that for $\varrho_j = \varrho(2^j\cdot)$ for a smooth function $\varrho$ with compact support in an annulus.

We recall an embedding for Besov spaces (see. i.e. [3, Proposition]).

**Theorem 4.2** (Besov embedding theorem)**.** *For any $1 \le p_1 \le p_2 \le \infty$ and $1 \le p_1 \le p_2 \le \infty$ the space $B^\alpha_{p_1,q_1}$ is continuously embedded in$B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}$. In other words,*

$$\|f\|_{B^\alpha_{p_2,q_2}} \le C\|f\|_{B^{\alpha+d(1/p_1-1/p_2)}_{p_1,q_1}},$$

*whenever the norms of $f$ are well-defined.*

In the course of our proof an alternative characterization of the spaces $B^\alpha_{p,p}$ (which, in our setup, are just fractional Sobolev spaces) turns out to be useful.

**Definition 4.3** (Sobolev-Slobodeckij norm)**.** *Let $\varphi \in \mathcal{S}'(\mathbb{T}^d)$. For every $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and for every $p \in [0,\infty]$ we define the the Sobolev-Slobodeckij norm as*

$$\|\varphi\|_{W^\alpha_p} := \|\varphi\|_{L^p} + \sum_{|m|=\lfloor\alpha\rfloor}\left(\int_{\mathbb{T}^d\times\mathbb{T}^d} \frac{|D^m\varphi(x) - D^m\varphi(y)|^p}{|x-y|^{d+(\alpha-\lfloor\alpha\rfloor)p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \in [0,\infty].$$

The space of functions for which the Sobolev-Slobodeckij norm is finite is called the fractional Sobolev space (see e.g. [31] Theorem 2.5.7 and the discussion in Section 2.2.2).

**Proposition 4.4.** *There exist constants a pair of constants $c, C > 0$ such that for $\phi \in \mathcal{S}'(\mathbb{T}^d)$*

$$c\|\varphi\|_{B^\alpha_{p,p}} \le \|\varphi\|_{W^\alpha_p} \le C\|\varphi\|_{B^\alpha_{p,p}}.$$

For time dependent functions taking values in a Banach space $\mathcal{X}$ the $\alpha$-Hölder norm (with $\alpha \in (0,1)$) is defined as

$$\|f\|_{C^\alpha \mathcal{X}} = \sup_{t\in[0,T]} \|f(t)\|_{\mathcal{X}} + \sup_{t,s\in[0,T]} \frac{\|f(t)-f(s)\|_{\mathcal{X}}}{|t-s|^\alpha}$$

In order to describe the spatio-temporal regularity of the processes under consideration we introduce the spaces

$$\mathcal{E}^\gamma \mathcal{C}^\alpha_p = \{f\colon (0,T] \to \mathcal{C}^\alpha_p \mid \|f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} = \sup_{t\in[0,T]} t^\gamma \|f(t)\|_{\mathcal{C}^\alpha_p} < \infty\},$$

and

$$\mathscr{L}^{\gamma,\alpha}_p = \{f \in \mathcal{E}^\gamma \mathcal{C}^\alpha_p \mid \|f\|_{\mathscr{L}^{\gamma,\alpha}_p} = \|f\|_{\mathcal{E}^\gamma \mathcal{C}^\alpha_p} + \|t \mapsto t^\gamma f(t)\|_{C^{\alpha/2} L^p} < \infty\}.$$

4.2. **Bounds on characteristic functions.** In this subsection we address different questions related to the regularity properties of characteristic functions and their Fourier transforms (in particular of balls and cubes). Restate the notation here? Probably not neccesary We recall the following facts about the Fourier transform of characteristic function of a ball as a Lemma.

**Lemma 4.5.** *Let $(D\varphi)_i = \frac{d\varphi}{dx_i}$ and $(D^2\varphi)_{i,j} = \frac{d^2\varphi}{dx_i dx_j}$ indicate the gradient and the Hessian matrix of a smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ respectively. Then*

$$\widehat{\chi}_{B_\varepsilon}(\cdot) = \widehat{\chi}(\varepsilon \cdot), \qquad D\widehat{\chi}(0) = 0, \qquad D^2\widehat{\chi}(0) = -(2\pi)^2 \nu_0 \mathrm{Id},$$

*with*

$$\nu_0 = \frac{1}{12} \quad in \quad d = 1, \qquad \nu_0 = \frac{1}{4\pi} \quad in \quad d = 2.$$

*In particular, for any choice of constants $c < 1 < C$, there exists a $\kappa(c,C)$ such that*

$$c \leq \frac{\vartheta_\varepsilon(k)}{-(2\pi)\nu_0 |k|^2} \leq C, \qquad \forall k\colon \ |k|\varepsilon \leq \kappa(c,C).$$

*Proof.* The scaling result follows from a change of variable. For the term involving the gradient, we have that for $i = 1, \dots, d$

$$(D\widehat{\chi})_i(0) = -2\pi\iota \fint_{B_1(0)} x_i e^{-2\pi\iota\langle k,x\rangle} \,\mathrm{d}x \Big|_{k=0} = 0.$$

For the term involving the Hessian, we observe that an analogous computation for $i \neq j$ shows that $(D^2\widehat{\chi})_{i,j}(0) = 0$ . If $i = j$ we find that

$$(D^2\widehat{\chi})_{i,i}(0) = -(2\pi)^2 \fint_{B_1(0)} \mathrm{d}x \ x_i^2 e^{-2\pi\iota\langle k,x\rangle} \Big|_{k=0} =: -(2\pi)^2 \nu_0,$$

with the value of $\nu_0$ as in the statement. The two-sided inequality follows by Taylor's Theorem. □

The following lemma is special case of well known results on the decay of characteristic function of a convex set. Since it plays an important role in the proofs, we shall provide the proof of this result for reader's convenience,

**Lemma 4.6.** *For any $\alpha \in \mathbb{N}_0$*

$$\left|D^\alpha \hat{\chi}_Q(k)\right| \lesssim_\alpha (1+|k|)^{-1}. \tag{18}$$

*Similarly,*

$$\left|D^\alpha \hat{\chi}_B(k)\right| \lesssim_\alpha (1+|k|)^{-\frac{d+1}{2}} \tag{19}$$

*Proof.* The Fourier transform of $\chi_Q$ is given by

$$\widehat{\chi}_Q(k) = \prod_{i=1}^{d} \left[ \frac{\sin(2\pi k_i/4)}{2\pi k_i/4} \right],$$

which immediately yields the estimate for $\alpha = 0$. The estimates for $\alpha \in \mathbb{N}_+$ follow by observing that the derivatives take form form $\sum_i \frac{1}{k^{\alpha_i}}(a_i \cos ck + b_i \sin ck)$ for $\alpha_i \geq 1$.

We turn out attention to $\chi_B$. Let $J_\nu(\cdot)$ be the Bessel function of the first kind with parameter $\nu$, that is

$$J_\nu(k) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{k}{2}\right)^{2m+\nu}$$

The Fourier transform of $\chi_Q$ is given by

$$\hat{\chi}_B(k) = c(d) \int_0^\pi \mathrm{d}t \ \sin^d(t) e^{-2\pi\iota|k|\cos(t)/4} \simeq |k|^{-d/2} J_{d/2}(\pi|k|/2) \tag{20}$$

Since $J_{\frac{1}{2}}(k) = \sqrt{\frac{2}{\pi k}} \sin k$, the bound for $d = 1$ is immediate.

For $d = 2$, we make use of the classical asymptotic bound for Bessel functions, which reads

$$\sup_{\varrho \geq 1} \varrho^{-1/2} |J_\nu(\varrho)| < +\infty.$$

We provide a simple proof of this bound in the next Lemma.

The bound for the derivatives follows by exploiting the identity (20) and a pair of identities

$$\begin{aligned} \partial_x J_n(x) = \frac{1}{2}(J_{n-1}(x) + J_{n+1}(x)), \qquad & \forall n \in \mathbb{Z}, \\ J_{-n}(\cdot) = (-1)^n J_n(\cdot) \qquad & \forall n \in \mathbb{N}_0. \end{aligned}$$

□

Since usually one requires a full asymptotic bound, we provide a simple proof for the estimate we require.

**Lemma 4.7.** *Fix $\nu \in \mathbb{R}$. Then*

$$\sup_{\varrho \geq 1} \varrho^{-1/2} |J_\nu(\varrho)| < +\infty,$$

*Proof.* We write the quantity of interest as

$$\int_{-1}^{1} \mathrm{d}x \ (1-x^2)^{\frac{d-1}{2}} e^{\iota\varrho x} = 2\mathrm{Re}\bigg( \int_0^1 \mathrm{d}x \ (1-x^2)^{\frac{d-1}{2}} e^{\iota\varrho x} \bigg).$$

A change variables $x = 1 - u^2$. yields

$$e^{i\varrho} \int_0^1 \mathrm{d}u \ \big(u^2(2-u^2)\big)^{\frac{d-1}{2}} e^{-\iota\varrho u^2} u = \frac{e^{i\varrho}}{\varrho^{\frac{d+1}{2}}} \int_0^{\sqrt{\varrho}} \mathrm{d}w \ \big(w^2(2-\frac{w^2}{\varrho})\big)^{\frac{d-1}{2}} e^{-\iota w^2} w$$

Observe that in order to obtain the desired bound it is now sufficient to show that the integral terms is bounded uniformly in $\rho$. After another change of variable $w = e^{-\iota\frac{\pi}{4}} z$ we obtain

$$\begin{aligned} &\int_0^{e^{\frac{\iota\pi}{4}}\sqrt{\varrho}} \mathrm{d}z \ \big(-\iota z^2(2+\iota z^2/\varrho)\big)^{\frac{d-1}{2}} e^{-z^2} z \\ &= \int_0^{\sqrt{\varrho}} \mathrm{d}z \ \big(-\iota z^2(2+\iota z^2/\varrho)\big)^{\frac{d-1}{2}} e^{-z^2} z + \int_0^{\pi/4} \mathrm{d}\varphi \ \big(-\iota\varrho e^{2\iota\varphi}(2+\iota e^{2\iota\varphi})\big)^{\frac{d-1}{2}} e^{-\varrho e^{2\iota\varphi}} \varrho e^{2\iota\varphi} \end{aligned}$$

The first integral can be trivially bounded uniformly over $\varrho$ while the second one is tends to 0 as $\rho$ tends to infinity as the exponential term dominates all the others. □

**Lemma 4.8.** *Let $f, g \in \mathcal{S}'(\mathbb{T}^d)$. For $p, q, r \in [1, \infty]$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$*

$$\|f * g\|_{C_r^{\alpha+\beta}} \lesssim \|f\|_{C_p^\alpha} \|g\|_{\mathcal{C}_q^\beta}.$$

*Proof.* By Young convolution inequality

$$\|\Delta_i(f * g)\|_{L^r} = \|\Delta_i f * \overline{\Delta}_i g\|_{L^r} \lesssim \|\Delta_i f\|_{L^p} \|\overline{\Delta}_i g\|_{L^q}, \tag{21}$$

where $\overline{\Delta}_i$ is associated with a dyadic partition of the unity different from the one we use for most of the proofs. Namely we require it satisfies $\{\overline{\varrho_j}\}_{j \geq -1}$ such that $\varrho_j \overline{\varrho_j} = \varrho_j$. Then the bound follows immediately, since the Besov norms associated to different dyadic partitions are equivalent (cf. [3, Remark 2.17]). □

The following lemma is a special case of of results obtained by [27]. The proof is included for completeness.

**Lemma 4.9** (Fractional regularity of characteristic function)**.** *Fix $\zeta \in (0, 1)$. Then, for $1 \leq p\infty$*

$$\|1_{B_\varepsilon}\|_{W_p^\zeta} < \infty.$$

*Proof.* We shall make use of the characterization of fractional Sobolev space in terms of Sobolev-Slobodeckij norm. Direct computation shows that

$$\|1_{B_\varepsilon}\|_{W_p^\zeta} = \|1_{B_\varepsilon}\|_{L^p} + \int_{\mathbb{T}^d \times \mathbb{T}^d} \frac{1_{B_\varepsilon(x)} - 1_{B_\varepsilon(y)}}{|x-y|^{d+\zeta p}} \mathrm{d}x\mathrm{d}y$$
$$= \|1_{B_\varepsilon}\|_{L^p} + 2 \int_{B_\varepsilon} \int_{\mathbb{T}^d \setminus B_\varepsilon} \frac{1_{B_\varepsilon(x)} - 1_{B_\varepsilon(y)}}{|x-y|^{d+\zeta p}} \mathrm{d}x\mathrm{d}y < \infty.$$

□

**Lemma 4.10.** *For $\zeta \in [0, 1)$ and $\alpha \in \mathbb{R}$*

$$\sup_{\varepsilon \leq 1} \varepsilon^\zeta \|\chi_\varepsilon * \varphi\|_{\mathcal{C}_p^{\alpha+\zeta}} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}.$$

*Proof.* By (21)

$$\|\Delta_j(\chi_\varepsilon * \varphi)\|_{L^p} \lesssim \|\overline{\Delta}_j \chi_\varepsilon\|_{L^1} \|\Delta_j \varphi\|_{L^p} \lesssim \|\overline{\Delta}_j \chi_\varepsilon\|_{L^1} 2^{-j\alpha} \|\varphi\|_{\mathcal{C}_p^\alpha}.$$

It is therefore sufficient to bound $\|\overline{\Delta}_j \chi_\varepsilon\|_{L^1}$. For that purpose we estimate

$$\|\overline{\Delta}_j \chi_\varepsilon\|_{L^1} = \|(2^j \varepsilon)^d \overline{K}(2^j \varepsilon \cdot) * \chi(\cdot)\|_{L^1} \lesssim (2^j \varepsilon \vee 1)^{-\zeta} \|\chi\|_{B_{1,\infty}^\zeta}$$

The last quantity is bounded by application of Lemma 4.9 and by observing that $1 \leq (2^j \varepsilon)^{-\zeta}$ whenever $2^j \varepsilon \leq 1$.

□

4.3. **Schauder estimates.** In this section we describe the gain of smoothness coming from the action of the semigroup which approximates the Anderson Hamiltonian.

Recall that by $\chi_\varepsilon(x)$ we denote the characteristic function of the ball of radius $\varepsilon$ centred at 0, that is

$$\chi_\varepsilon(x) = \varepsilon^{-d} 1_{\{B_\varepsilon(0)\}}(x)$$

and that the operator $\mathcal{A}_\varepsilon$ is defined by

$$\mathcal{A}_\varepsilon \varphi(x) = \varepsilon^{-2} \bigg( \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(y)} \varphi(z) \, \mathrm{d}z \, \mathrm{d}y - \varphi(x) \bigg) = \varepsilon^{-2} \big[ \Pi_\varepsilon^2 \varphi - \varphi \big](x). \tag{22}$$

The operator $\mathcal{A}_\varepsilon^e$ is a Fourier multiplier on $\mathcal{S}'(\mathbb{T}^d)$, that is

$$\mathcal{A}_\varepsilon^e \varphi = \vartheta_\varepsilon(D) \varphi,$$

where, as before,

$$\vartheta_\varepsilon(\cdot) = \varepsilon^{-2}(\widehat{\chi_{B_\varepsilon}}^2(\cdot) - 1).$$

Therefore the naturally defined functional calculus holds and hence $e^{t\mathcal{A}_\varepsilon}$ is well-defined.

We investigate what is the regularity of $\int_0^t \mathrm{d}s\, e^{(t-s)\mathcal{A}_\varepsilon}\varphi$ in comparison to regularity of $\varphi$ on the Besov scale.

It will be convenient to consider the action of $\mathcal{A}_\varepsilon$ for large and small Fourier modes separately. Lemma 4.11 provides the correct choice for this division.

**Lemma 4.11.** *Fix $p \in [1, \infty]$. There exist a constant $\kappa_0 > 0$, independent of $p$, and a pair of constants $C, c > 0$ such that for every $\varphi \in \mathcal{S}'(\mathbb{T}^d), \varepsilon \in (0, 1], t \geq 0, \alpha \in \mathbb{R}$ and $j \geq -1$*

$$\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)} \leq Ce^{-ct2^{2j}}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for } \; 2^j\varepsilon \leq \kappa_0, \tag{23}$$

$$\|\Delta_j e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)} \leq Ce^{-ct\varepsilon^{-2}}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for } \; 2^j\varepsilon > \kappa_0. \tag{24}$$

*Similarly,*

$$\|\Delta_j \mathcal{A}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)} \lesssim 2^{-(\alpha-2)j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{if } \; 2^j\varepsilon \leq \kappa_0, \tag{25}$$

$$\|\Delta_j \mathcal{A}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)} \lesssim \varepsilon^{-2}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{if } 2^j\varepsilon > \kappa_0. \tag{26}$$

*Proof.* Since all of the estimates follow the same pattern, in the course of the proof we shall focus our attention on inequalities (23) and (24). The proof is divided into several steps. We begin by restating the inequalities for distributions on $\mathbb{R}^d$. Afterwards we examine the behaviour on large and small scales separately. The precise separation of modes is chosen based on Lemma 4.5. We conclude by commenting briefly on how to adapt the previous calculation for (25) and (26).

*Step 0.* It will be convenient to restate the problem on $\mathbb{R}^d$. This simplifies the discussion, since the latter space is invariant under scaling. So the operator $\mathcal{A}_\varepsilon$ is defined only on $\mathcal{S}'(\mathbb{T}^d)$, but since $\mathcal{A}_\varepsilon$ acts locally, equation (22) defines an operator on $\mathcal{S}'(\mathbb{R}^d)$ as well. We shall denote it by $\mathcal{A}^e_\varepsilon$. Let $\pi\colon \mathcal{S}'(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote a periodic extension of distribution on $\mathbb{T}^d$ to full space. The adjoint operator $\pi^*\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{T}^d)$ is given by

$$\pi^*\varphi(\cdot) = \sum_{k\in\mathbb{Z}^d} \varphi(\cdot + k).$$

It is well defined for elements of the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. We observe that $\pi(\mathcal{A}_\varepsilon\varphi) = \mathcal{A}^e_\varepsilon\pi(\varphi)$. Similarly, by Poisson summation formula (see Lemma 4.1), $\pi(\Delta_j\varphi) = \Delta_j\pi(\varphi)$.

As a consequence of this last observation we note that for any $a > \frac{d}{p}$

$$\|\Delta_j\pi(\varphi)\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \simeq_{a,p} \|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)},$$

where $\|f\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} = \|f(\cdot)/(1+|\cdot|^2)^{\frac{a}{2}}\|_{L^p(\mathbb{R}^d)}$. Therefore in order to show 23 and (24) it is sufficient to show that for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, there exists an $a > \frac{d}{p}$ such that

$$\|\Delta_j e^{t\mathcal{A}^e_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \leq Ce^{-ct2^{2j}}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)}, \quad \text{for } \; 2^j\varepsilon \leq \kappa_0$$
$$\|\Delta_j e^{t\mathcal{A}^e_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \leq Ce^{-ct\varepsilon^{-2}}2^{-\alpha j}\|\varphi\|_{\mathcal{C}^\alpha_p}, \quad \text{for } \; 2^j\varepsilon > \kappa_0.$$

Analogous bounds hold for 25 and (26).

*Step 1.* From now on all functions and operators are defined on $\mathbb{R}^d$, and for simplicity write $\widehat{f} = \mathcal{F}_{\mathbb{R}^d}f$. Let $\psi$ be a radial function with compact support such that $\rho\psi = \rho$. By Young's inequality for convolution

$$\|\Delta_j e^{t\mathcal{A}^e_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \lesssim \|\mathcal{F}^{-1}_{\mathbb{R}^d}(e^{t\vartheta_\varepsilon(\cdot)}\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\cdot\rangle^{-a})}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)}. \tag{27}$$

*Step 2.* We begin with the estimate on the large scales. We observe that one can choose a constant $m$ such that

$$\|\mathcal{F}^{-1}_{\mathbb{R}^d}(e^{t\vartheta_\varepsilon(\cdot)}\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\cdot\rangle^{-a})} \lesssim \|\mathcal{F}^{-1}_{\mathbb{R}^d}[(1+\Delta^m)e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)]\|_{L^\infty(\mathbb{R}^d)}. \tag{28}$$

Now, in view of Lemma 4.5 there exists $\kappa_0 > 0$ and a pair of constants $0 < c_1 \leq 1 \leq c_2$ such that for $2^j\varepsilon \leq \kappa_0$

$$c \leq \sup_{k\in\mathrm{supp}(\psi)} \frac{\vartheta_\varepsilon(2^j k)}{-(2\pi)^2\nu_0 2^{2j}|k|^2} \leq C.$$

To bound the term involving derivatives we observe that:

$$D[t\vartheta_\varepsilon(2^j\cdot)](k) = f(2^j\varepsilon k)t2^{2j}|k|, \qquad f(k) = 2\widehat{\chi}(k)\frac{D\widehat{\chi}(k)}{|k|}$$

where $f$ is smooth on $\mathbb{R}^d$. Hence, there exists a constant $\bar{c} > c > 0$:

$$\sup_{k\in\mathrm{supp}(\psi)} |(1+\Delta^m)e^{t\vartheta_\varepsilon(2^j k)}| \lesssim e^{-\bar{c}(2\pi)^2\nu_0 t2^{2j}}(1+t2^{2j})^m \lesssim e^{-c(t2^{2j})}.$$

This concludes the proof for (23) and (24).

*Step 3.* We turn out attention to (24). For $2^j\varepsilon > \kappa$, a tighter control on $\widehat{\chi}(k)$ is required, at least for large $k$. Since $\chi$ is not smooth, the decay at infinity is not faster than any polynomial. We begin directly with Equation (27). Set $a = d+1$ and changing variables, note that

$$\begin{aligned}
\|\mathcal{F}^{-1}(e^{t\vartheta_\varepsilon(\cdot)}\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\rangle^{-a})} &= \|\langle 2^{-j}\cdot\rangle^a\mathcal{F}^{-1}[e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)]\|_{L^1(\mathbb{R}^d)} \\
&\lesssim \int_{\mathbb{R}^d} \mathrm{d}x \; \frac{1}{1+\sum_{i=1}^d |x_i|^{d+1}}\left(1+\sum_{i=1}^d 2^{-j(d+1)}|x_i|^{2(d+1)}\right)\left|\mathcal{F}^{-1}[e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)](x)\right| \\
&\leq \|e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^1}+\sum_{i=1}^d 2^{-j(d+1)}\|\partial_{k_i}^{2(d+1)}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^1}.
\end{aligned}$$

Since $|\hat{\chi}(k)| < 1$ for $k \neq 0$, there exists $c_3 > 0$ such that

$$\vartheta_\varepsilon(2^j k) \leq -c\varepsilon^{-2},$$

This is sufficient to show that

$$\|e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^1} \lesssim e^{-ct\varepsilon^{-2}}. \tag{29}$$

In order to tackle the terms which involve derivatives first observe that

$$\partial_{k_i}^{2(d+1)}e^{t\vartheta_\varepsilon(2^j k)} = \partial_{k_i}^{2d+1}\left[e^{t\vartheta_\varepsilon(2^j k)}2\hat{\chi}(2^j\varepsilon k)[\partial_{k_i}\hat{\chi}](2^j\varepsilon k)\right]\cdot(2^j\varepsilon)\cdot(t\varepsilon^{-2}).$$

Let $\{m_\ell\}_{\ell=1,\dots,2(d+1)}$ be a sequence such that $\sum_{\ell=1}^{2(d+1)}\ell\cdot m_\ell = 2(d+1)$. Iterating the above procedure, we apply Faá Di Bruno's formula to obtain

$$2^{-j(d+1)}\partial_{k_i}^{d+1}e^{t\vartheta_\varepsilon(2^j k)} = 2^{-j(d+1)}e^{t\vartheta_\varepsilon(2^j k)}\sum_{\{m_\ell\}}\prod_{\ell=1}^{2(d+1)}\left(\partial_{k_i}^{\ell-1}\left[2\hat{\chi}(\cdot)[\partial_{k_i}\hat{\chi}(\cdot)]\right]\big|_{2^j\varepsilon k}\cdot(2^j\varepsilon)^\ell\right)^{m_\ell}\cdot(t\varepsilon^{-2})^{m_\ell}.$$

In view of Lemma 4.6

$$\sup_{k\in\mathrm{supp}(\psi)}|\partial_{k_i}^{\ell-1}\left[2\hat{\chi}(\cdot)[\partial_{k_i}\hat{\chi}(\cdot)]\right]\big|_{2^j\varepsilon k}| \lesssim \frac{1}{1+|2^j\varepsilon|^{d+1}}.$$

Since $e^{-\hat{c}x^2}x^m \lesssim e^{-cx^2}$ for $c \leq \hat{c}$,

$$\|\partial_{k_i}^{2(d+1)}e^{t\vartheta_\varepsilon(2^j\cdot)}\psi(\cdot)\|_{L^1} \lesssim e^{-ct\varepsilon^{-2}}2^{-j(d+1)}(2^j\varepsilon)^{2(d+1)}\sum_{\{m_\ell\}}\prod_{\ell=1}^{2(d+1)}\langle 2^j\varepsilon\rangle^{-m_\ell(d+1)} \lesssim e^{-ct\varepsilon^{-2}},$$

since at least one of the elements of the sequence $m_\ell$ is positive. Combining this estimates provides (24).

*Step 4.* We now briefly address (25) and (26). As before, in the spirit of (27), observe that

$$\|\Delta_j\mathcal{A}_\varepsilon^e\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)} \lesssim \|\mathcal{F}_{\mathbb{R}^d}^{-1}(\vartheta_\varepsilon(\cdot)\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\cdot\rangle^{-a})}\|\Delta_j\varphi\|_{L^p(\mathbb{R}^d,\langle\cdot\rangle^a)}.$$

As in Step 2, if $2^j\varepsilon \le \kappa$,

$$\|\mathcal{F}^{-1}_{\mathbb{R}^d}(\vartheta_\varepsilon(\cdot)\psi(2^{-j}\cdot))\|_{L^1(\mathbb{R}^d,\langle\cdot\rangle^{-a})} \lesssim \|\langle 2^{-j}\cdot\rangle^a \mathcal{F}^{-1}_{\mathbb{R}^d}(\vartheta_\varepsilon(2^j\cdot)\psi(\cdot))\|_{L^1(\mathbb{R}^d)},$$

so that to obtain the bounds which lead to (25) it is sufficient to follow the calculations in Step 2.

To show (26), one can follow the calculations in Step 3 verbatim, which leads to the the term of the form

$$\|\vartheta_\varepsilon(2^j\cdot)\psi(\cdot)\|_{L^1} + \sum_{i=1}^d 2^{-j(d+1)}\|\partial^{2(d+1)}_{k_i}\vartheta_\varepsilon(2^j\cdot)\|_{L^1}.$$

By analogous calculations, this term is bounded by $C\varepsilon^{-2}$.

□

In view of Lemma 4.11, it is natural to define a cut-off operator which separates large and small Fourier modes. The appropriate value point of cut-off is given by (28).

**Definition 4.12.** *Let* $\urcorner\colon \mathbb{R}^d \to \mathbb{R}$ *be a smooth radial function with compact support which is constant outside of an annulus* $A^R_r = \{x \in \mathbb{R}^d \colon r \le |x| \le R\}$ *for some* $0 < r < R$ *and such that:*

$$\kappa_0 \in (r, R), \qquad \urcorner(x) = 1,\ \forall x \in A^r_0, \qquad \urcorner(x) = 0,\ \forall x \in A^\infty_R,$$

*with* $\kappa_0$ *chosen as in* (28). *Define*

$$\mathcal{P}_\varepsilon = \urcorner(\varepsilon D), \qquad \mathcal{Q}_\varepsilon = (1-\urcorner)(\varepsilon D).$$

For a function $\phi$, we shall sometimes refer to $\mathcal{P}_\varepsilon\phi$ as the large scales of $\phi$ and to $\mathcal{Q}_\varepsilon\phi$ as small scales of $\phi$. The next lemma shows that the cut-off operators are bounded.

**Lemma 4.13.** *Consider* $\alpha \in \mathbb{R}$ *and* $p \in [1,\infty]$. *For* $\urcorner$ *as in Definition 4.12 one can bound uniformly over* $\varepsilon \in (0,1)$*:*

$$\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}, \qquad \|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

*Proof.* Let $\widehat{\urcorner}(x) = \mathcal{F}^{-1}_{\mathbb{R}^d}\urcorner(x)$. By an application of the Poisson summation formula (see Lemma 4.1)

$$\begin{aligned}\|\urcorner(\varepsilon D)\varphi\|_{\mathcal{C}^\alpha_p} &= \sup_{j\ge -1} 2^{j\alpha}\|(\mathcal{F}^{-1}_{\mathbb{T}^d}[\urcorner(\varepsilon\cdot)]) * \Delta_j\varphi\|_{L^\infty} \lesssim \|\mathcal{F}^{-1}_{\mathbb{T}^d}[\urcorner(\varepsilon\cdot)]\|_{L^1(\mathbb{T}^d)}\|\varphi\|_{\mathcal{C}^\alpha_p}\\ &\lesssim \|\varepsilon^{-d}\widehat{\urcorner}(\varepsilon^{-1}\cdot)\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.\end{aligned}$$

The same argument shows that $(1-\urcorner(a\cdot))$ is bounded as well. □

Next Lemma demonstrates that $\mathcal{A}_\varepsilon$ approximates the Laplace operator.

**Lemma 4.14.** *For any* $\alpha \in \mathbb{R}, \zeta > 0, p \in [1,\infty]$ *and* $\varphi \in \mathcal{C}^\alpha_p$

$$\|\mathcal{A}_\varepsilon\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^{\alpha-2}_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

*Moreover, as* $\varepsilon \to 0$

$$\mathcal{A}_\varepsilon\varphi \to \nu_0\Delta\varphi \quad in \quad \mathcal{C}^{\alpha-2-\zeta}_p,$$

*where*

$$\nu_0 = \frac{1}{12} \quad for \quad d = 1, \qquad \nu_0 = \frac{1}{4\pi} \quad for \quad d = 2.$$

*Proof.* First observe that small scales are negligible. Indeed:

$$\|\mathcal{Q}_\varepsilon\mathcal{A}_\varepsilon\varphi\|_{\mathcal{C}^{\alpha-2-\zeta}_p} \lesssim \varepsilon^{-2}\sup_{j\gtrsim\varepsilon^{-1}} 2^{j(\alpha-2-\zeta)}\|\Delta_j\varphi\|_{L^p} \lesssim \varepsilon^\zeta\|\varphi\|_{\mathcal{C}^\alpha_p},$$

which tends to 0 as $\varepsilon$ tends to 0. On large scales, Lemma 4.11 implies that

$$\sup_{\varepsilon\in(0,1)} \|\mathcal{A}_\varepsilon\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}^{\alpha-2}_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

Combining those two observations provides the first bound and guarantees compactness in $\mathcal{C}_p^{\alpha-2-\zeta}$. Convergence follows since, by Lemma 4.5,

$$\mathcal{F}_{\mathbb{T}^d}[\mathcal{A}_\varepsilon\mathcal{P}_\varepsilon\varphi](k) = \urcorner(\varepsilon k)\frac{\hat{\chi}^2(\varepsilon k)-1}{\varepsilon^2}\hat{\varphi}(k) \to -(2\pi)^2\nu_0|k|^2\hat{\varphi}(k) = \mathcal{F}_{\mathbb{T}^d}[\nu_0\Delta\varphi](k).$$

□

The regularity gain provided by the operator $\mathcal{A}_\varepsilon$ can be described by the means of elliptic Schauder estimates. not corrected. I also changed the statement from $B^\alpha_{p,q}$ to $\mathcal{C}^\alpha_p$.. The convergence seems wrong! I wrote the one I think should hold: we still have to add it to the proof.

**Lemma 4.15.** *Fix any $\alpha \in \mathbb{R}$, $\delta \in [0,1]$ and $p \in [1,\infty]$. Uniformly over $\lambda > 1, \varepsilon \in (0,1/2)$ and $\varphi \in \mathcal{C}^\alpha_p$ the following estimates hold:*

$$\lambda^{-\delta}\|\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{\mathcal{C}_p^{\alpha+2(1-\delta)}} + \lambda^{-\delta}\varepsilon^{-2(1-\delta)}\|\mathcal{Q}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{\mathcal{C}^\alpha_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha_p}.$$

*Moreover, as $\varepsilon$ tends to 0,*

$$\mathcal{P}_\varepsilon\big(-\mathcal{A}_\varepsilon-\lambda\big)^{-1}\varphi \to (\nu_0\Delta-\lambda)^{-1}\varphi$$

*where convergence is in $\mathcal{C}_p^{\alpha-2-\zeta}$ for any $\zeta>0$ and $\nu_0$ is as in Lemma 4.14.*

*Proof.* Start with the smallscale bound. One can write:

$$\begin{aligned}
\|\Delta_j\mathcal{Q}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{L^p(\mathbb{T}^d)} &\le \|\mathcal{F}^{-1}_{\mathbb{T}^d}[(-\vartheta_\varepsilon+\lambda)^{-1}(1-\urcorner)(\varepsilon\cdot)]\|_{L^1(\mathbb{T}^d)}\|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)}\\
&\lesssim \varepsilon^2\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(-\hat{\chi}^2+1+\varepsilon^2\lambda)^{-1}(1-\urcorner)(\varepsilon\cdot)]\|_{L^1(\mathbb{R}^d)}\|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)}\\
&= \varepsilon^2\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(-\hat{\chi}^2+1+\varepsilon^2\lambda)^{-1}(1-\urcorner)]\|_{L^1}\|\Delta_j\varphi\|_{L^p}\\
&\le \varepsilon^2\|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)}\Big[(1+\varepsilon^2\lambda)^{-1}\\
&\qquad + \|\mathcal{F}^{-1}_{\mathbb{R}^d}[(-\hat{\chi}^2+1+\varepsilon^2\lambda)^{-1}(1-\urcorner)-(1+\varepsilon^2\lambda)^{-1}]\|_{L^1(\mathbb{R}^d)}\Big].
\end{aligned}$$

through the Poisson summation formula and scaling. Now the first term can be estimated through:

$$\frac{1}{1+\varepsilon^2\lambda} \le \varepsilon^{-2\delta}\lambda^{-\delta}$$

while the second term is bounded, due to the compact support of $\urcorner$, by:

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|\frac{1-\urcorner(k)}{-\hat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\right| + \sum_{i=1}^d\left|\partial_i^{2d}\left(\frac{1-\urcorner(k)}{-\hat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\right)\right|\mathrm{d}k\\
\lesssim \frac{1}{1+\varepsilon^2\lambda} + \int_{\mathbb{R}^d}\sum_{i=1}^d\left|\partial_i^{2d}\left(\frac{1}{-\hat{\chi}^2(k)+1+\varepsilon^2\lambda}-\frac{1}{1+\varepsilon^2\lambda}\right)\right|\mathrm{d}k\\
= \frac{1}{1+\varepsilon^2\lambda}\left(1+\int_{\mathbb{R}^d}\sum_{i=1}^d\left|\partial_i^{2d}\left(\frac{-\hat{\chi}^2(k)}{-\hat{\chi}^2(k)+1+\varepsilon^2\lambda}\right)\right|\mathrm{d}k\right)\\
\lesssim \frac{1}{1+\varepsilon^2\lambda} \le \varepsilon^{-2\delta}\lambda^{-\delta},
\end{aligned}$$

where we have used that $\chi$ is smooth and decays rapidly for $k\to\infty$. here change for characteristic function.

Now let us pass to large scales. For $j$ such that $2^j\varepsilon \lesssim 1$ one has, applying as before the Poisson summation formula:

$$\|\Delta_j\mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon+\lambda)^{-1}\varphi\|_{L^p(\mathbb{T}^d)} \le \|\mathcal{F}^{-1}_{\mathbb{T}^d}[\urcorner(\varepsilon\cdot)\varrho_j(\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)]\|_{L^1(\mathbb{R}^d)}\|\Delta_j\varphi\|_{L^p(\mathbb{T}^d)},$$

with $\overline{\varrho}$ defined as in Equation (**??**). Then estimate:

$$\begin{aligned}\|\mathcal{F}^{-1}_{\mathbb{R}^d}[\urcorner(\varepsilon\cdot)\overline{\varrho}_j(\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)]\|_{L^1(\mathbb{R}^d)} &\lesssim \|\mathcal{F}^{-1}_{\mathbb{R}^d}[\overline{\varrho}_j(-\vartheta_\varepsilon+\lambda)^{-1}]\|_{L^1(\mathbb{R}^d)}\\ &\lesssim \|(1+\sum_{i=1}^d |x_i|^{2d})\mathcal{F}^{-1}_{\mathbb{R}^d}[\overline{\varrho}_0(-\vartheta_\varepsilon+\lambda)^{-1}(2^j\cdot)](x)\|_{L^\infty(\mathbb{R}^d)}\\ &\lesssim \sup_{k\in\mathrm{supp}(\varrho_0)}\Big(|(-\vartheta_\varepsilon+\lambda)^{-1}(2^jk)|+\sum_{i=1}^d |\partial_i^{2d}(-\vartheta_\varepsilon+\lambda)^{-1}(2^j\cdot)|\big|_k\Big).\end{aligned}$$

Following the calculations in Step 2 of the proof of Proposition 4.16 one can estimate this via:

$$\frac{1}{2^{2j}+\lambda}\lesssim 2^{-2j(1-\delta)}\lambda^{-\delta}.$$

This proves the required bound. ☐

We may now state the main result of this section, the parabolic Schauder estimates. Recall that $T$, which appears in the statement of the estimate is implicit in the definition of the space $\mathscr{L}_p^{\gamma,\alpha+2}$.

**Proposition 4.16.** *Fix $p\in[1,\infty], T_f>0, \gamma\in[0,1)$ and $\alpha\in(-2,0), \beta\in[\alpha,\alpha+2)\cap(0,2)$. For any $\varphi\in\mathcal{C}_p^\alpha$ and $f\in\mathcal{E}_T^\gamma\mathcal{C}_p^\alpha$, and all $T\in[0,T_f]$*

$$\|t\mapsto \mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathscr{L}_p^{(\beta-\alpha)/2,\beta}}\lesssim\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}, \tag{30}$$

$$\Big\|t\mapsto\int_0^t\mathcal{P}_\varepsilon e^{(t-s)\mathcal{A}_\varepsilon}f(s)\,\mathrm{d}s\Big\|_{\mathscr{L}_p^{\gamma,\alpha+2}}\lesssim\|\mathcal{P}_\varepsilon f\|_{\mathcal{E}_T^\gamma\mathcal{C}_p^\alpha}, \tag{31}$$

*with constants independent of $f,\varphi,T$.*

*In addition, let $\zeta_1,\zeta_2\in[0,1)$ such that $\zeta_1+\zeta_2<1$ and $\delta_1,\delta_2,\delta_3\in[0,1]$ such that $\delta_1+\delta_2+\delta_3=1$. Then*

$$\|t\mapsto t^{\zeta_1+\zeta_2}\mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{C^{\zeta_1}\mathcal{C}_p^\alpha}\lesssim\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}, \tag{32}$$

$$\Big\|t\mapsto t^\gamma\int_0^t e^{(t-s)\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon f(s)\,\mathrm{d}s\Big\|_{C^{\delta_1}\mathcal{C}_p^\alpha}\lesssim\varepsilon^{2\delta_2}T^{\delta_3}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}_p^\alpha}. \tag{33}$$

*with constants independent of $f,\varphi,T$.*

*Proof. Step 1.* We begin by investigating the regularity gain on the large scales. With intention to prove (30), by (23)

$$\begin{aligned}\sup_{j\ge-1}2^{\beta j}\|\Delta_j\mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{T}^d)}&\lesssim\sup_{j\ge-1}e^{-ct2^{2j}}2^{(\beta-\alpha)j}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\\ &=t^{-\frac{\beta-\alpha}{2}}\sup_{j\ge-1}e^{-ct2^{2j}}(t2^{2j})^{\frac{\beta-\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\lesssim t^{-\frac{\beta-\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}.\end{aligned}$$

Therefore

$$\|t\mapsto\mathcal{P}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathcal{E}^{(\beta-\alpha)/2}\mathcal{C}_p^\beta}\lesssim\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha},$$

Similarly, for (31)

$$\sup_{j\ge-1}2^{j(\alpha+2)}\Big\|\int_0^t\Delta_j e^{(t-s)\mathcal{A}_\varepsilon}\mathcal{P}_\varepsilon f(s)\,\mathrm{d}s\Big\|_{L^p(\mathbb{T}^d)}\lesssim\|\mathcal{P}_\varepsilon f\|_{\mathcal{E}_T^\gamma\mathcal{C}_p^\alpha}\sup_{j\ge-1}2^{j2}\int_0^t e^{-cs2^{2j}}(t-s)^{-\gamma}\,\mathrm{d}s.$$

which can be bounded by $\|\mathcal{P}_\varepsilon f\|_{\mathcal{E}_T^\gamma\mathcal{C}_p^\alpha}$ by the same arguments as in the proof of [15, Lemma A.9]. We still need to address the temporal regularity for both terms. A direct application of Lemma 4.11 leads to

$$\|(e^{t\mathcal{A}_\varepsilon}-\mathrm{Id})\mathcal{P}_\varepsilon\varphi\|_{L^p(\mathbb{T}^d)}=\Big\|\int_0^t e^{s\mathcal{A}_\varepsilon}\mathcal{A}_\varepsilon\mathcal{P}_\varepsilon\varphi\,\mathrm{d}s\Big\|_{L^p(\mathbb{T}^d)}\lesssim\int_0^t s^{-1+\frac{\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\,\mathrm{d}s\simeq t^{\frac{\alpha}{2}}\|\mathcal{P}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}. \tag{34}$$

To conclude the proof of both (30) and (31) it is now sufficient to follow the same ideas as in [16, Lemma 6.6].

*Step 2.* We turn our attention to the bounds for small scales which will lead us to proofs of (32) and (33). For that purpose we fix $\zeta_1 = \delta_1 = 0$ first. With calculations in the same spirit as in the Step 1, we arrive at

$$\|\mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{\mathcal{C}_p^\alpha} = \sup_{j\geq -1} 2^{\alpha j}\|\Delta_j \mathcal{Q}_\varepsilon e^{t\mathcal{A}_\varepsilon}\varphi\|_{L^p(\mathbb{R}^d)} \lesssim e^{-ct\varepsilon^{-2}}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} \lesssim (t\varepsilon^{-2})^{-\delta}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}.$$

For the inequality (33), if $\delta_3 > 0$ the spatial bound follows from the previous result. If $\delta_3 = 0$, we observe that

$$\Big\|\int_0^t \mathcal{Q}_\varepsilon e^{(t-s)\mathcal{A}_\varepsilon} f(s)\,\mathrm{d}s\Big\|_{\mathcal{C}_p^\alpha} \lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma \mathcal{C}_p^\alpha}\int_0^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim \varepsilon^2 t^{-\gamma}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}_p^\alpha}.$$

The last bound in the above inequality is obtained in the same spirit as [15, Lemma A.9]. Namely, choose $\lambda \in (0, t/2)$ and split the integral at time $\lambda$. We note that

$$\int_0^\lambda e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \leq \int_0^\lambda (t-s)^{-\gamma}\,\mathrm{d}s = t^{-\gamma+1}\int_0^{\lambda/t}(1-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{-\gamma}\lambda,$$

since, as $\lambda/t \leq 1/2$, $1-(1-\lambda/t)^{(1-\gamma)} \lesssim \lambda/t$. W then observe that for any $\varrho \in (0,1)$,

$$\begin{aligned}\int_\lambda^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim \int_\lambda^t (s\varepsilon^{-2})^{-(1+\varrho)}(t-s)^{-\gamma}\,\mathrm{d}s &\lesssim t^{-\gamma-\varrho}\varepsilon^{2(1+\varrho)}\int_{\lambda/t}^1 s^{-(1+\varrho)}(1-s)^{-\gamma}\,\mathrm{d}s\\ &\lesssim t^{-\gamma}\varepsilon^{2(1+\varrho)}\lambda^{-\varrho}.\end{aligned}$$

If $\varepsilon^2 \leq t/2$, choosing $\lambda = \varepsilon^2$ provides the result. Otherwise, one simply notes that

$$\int_0^t e^{-cs\varepsilon^{-2}}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{1-\gamma} \lesssim t^{-\gamma}\varepsilon^2.$$

*Step 3.* We now investigate the full temporal regularity for (32) and (33), that is, we allow for $\zeta_1, \delta_1 > 0$. We first observe that for $\delta \in [0,1)$

(35)
$$\|(e^{t\mathcal{A}_\varepsilon}-\mathrm{Id})\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} = \Big\|\int_0^t e^{s\mathcal{A}_\varepsilon}\mathcal{A}_\varepsilon\mathcal{Q}_\varepsilon\varphi\,\mathrm{d}s\Big\|_{\mathcal{C}_p^\alpha} \lesssim \|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\int_0^t (s\varepsilon^{-2})^{-\delta}\varepsilon^{-2}\,\mathrm{d}s = \|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\varepsilon^{2(\delta-1)}t^{1-\delta}.$$

For $\delta \in [0,1)$ and $\zeta = \zeta_1+\zeta_2 \in [0,1)$, the temporal regularity of the first terms can be established via

$$\begin{aligned}\|t^\zeta e^{t\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi - s^\zeta e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} &\lesssim (t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} + s^\zeta\|(e^{(t-s)\mathcal{A}_\varepsilon}-\mathrm{Id})e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\\ &\lesssim (t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} + s^\zeta(t-s)^{1-\delta}\varepsilon^{2(\delta-1)}\|e^{s\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha}\\ &\lesssim [(t^\zeta-s^\zeta)t^{-\zeta_2}\varepsilon^{2\zeta_2} + (t-s)^{1-\delta}\varepsilon^{2(\delta-1)}\varepsilon^{2\zeta}]\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha} \lesssim (t-s)^{\zeta_1}\varepsilon^{2\zeta_2}\|\mathcal{Q}_\varepsilon\varphi\|_{\mathcal{C}_p^\alpha},\end{aligned}$$

where in the last step we set $\delta = 1-\zeta_1$ and notice that $(t^\zeta-s^\zeta)t^{-\zeta_2} \lesssim (t-s)^{\zeta_1}$.

The bound for (33) follows similar pattern. For simplicity write $V(t) = \int_0^t e^{(t-s)\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon f(s)\,\mathrm{d}s$. Then

$$\|t^\gamma V(t) - s^\gamma V(s)\|_{\mathcal{C}_p^\alpha} \leq (t^\gamma - s^\gamma)\|V(t)\|_{\mathcal{C}_p^\alpha} + s^\gamma\Big\|\int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon f(r)\,\mathrm{d}r\Big\|_{\mathcal{C}_p^\alpha} + s^\gamma\|(e^{(t-s)\mathcal{A}_\varepsilon}-\mathrm{Id})V(s)\|_{\mathcal{C}_p^\alpha}.$$

The only term for which the estimation does not follow the already established pattern is the middle one, for which we observe that

$$
\begin{aligned}
s^{\gamma}\Big\| \int_s^t e^{(t-r)\mathcal{A}_\varepsilon}\mathcal{Q}_\varepsilon f(r)\,\mathrm{d}r\Big\|_{\mathcal{C}^\alpha_p} &\lesssim s^{\gamma}\int_s^t ((t-r)\varepsilon^{-2})^{-\delta_2} r^{-\gamma}\,\mathrm{d}r \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p} \\
&\lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\varepsilon^{2\delta_2}s^{\gamma}t^{-\delta_2-\gamma+1}\int_{s/t}^1 (1-r)^{-\delta_2}r^{-\gamma}\,\mathrm{d}r \lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\varepsilon^{2\delta_2}t^{1-\delta_2}\int_{s/t}^1 (1-r)^{-\delta_2}\,\mathrm{d}r \\
&\lesssim \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\varepsilon^{2\delta_2}t^{1-\delta_2}(1-s/t)^{1-\delta_2} \le \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\varepsilon^{2\delta_2}(t-s)^{1-\delta_2} \le \|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}\varepsilon^{2\delta_2}T^{\delta_3}(t-s)^{\delta_1},
\end{aligned}
$$

which completes the proof of the proposition.

Brief computation as to why: $(t^\gamma - s^\gamma)t^{-\delta} \lesssim (t-s)^{\gamma-\delta}$, provieded $\gamma-\delta \in (0,1)$. We compute for $s \le t/2$:

$$t^{\gamma}(1-(s/t)^{\gamma})t^{-\delta} \lesssim t^{\gamma-\delta} \lesssim (t-s)^{\gamma-\delta}+s^{\gamma-\delta} \lesssim (t-s)^{\gamma-\delta}.$$

And for $s \ge t/2$:

$$t^{\gamma-\delta}(1-(1-(\frac{t-s}{t}))^{\gamma}) \lesssim t^{\gamma-\delta}(1-(1-(\frac{t-s}{t}))^{\gamma})^{\gamma-\delta} \lesssim (t-s)^{\gamma-\delta}$$

Brief calculation for the first term:

$$
\begin{aligned}
&(t^\gamma - s^\gamma)\|V(t)\|_{\mathcal{C}^\alpha_p} \lesssim (t-s)^{\gamma+\delta}t^{-\delta}\|V(t)\|_{\mathcal{C}^\alpha_p} \lesssim (t-s)^{\gamma+\delta}t^{-\delta}t^{-\gamma}\varepsilon^{2\delta_2}t^{1-\delta_2}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p} \\
&= (t-s)^{\delta_1}\varepsilon^{2\delta_2}t^{1-\delta_1-\delta_2}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}
\end{aligned}
$$

for $\delta = \delta_1 - \zeta$.

Brief calculation for the last term:

$$
\begin{aligned}
s^{\gamma}\|(e^{(t-s)\mathcal{A}_\varepsilon} - \mathrm{Id})V(s)\|_{\mathcal{C}^\alpha_p} &\lesssim s^{\gamma}\varepsilon^{2(\delta-1)}(t-s)^{1-\delta}\|V(s)\|_{\mathcal{C}^\alpha_p} \lesssim \varepsilon^{2(\delta-1)}(t-s)^{1-\delta}\varepsilon^{2\delta'}s^{1-\delta'}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p} \\
&\lesssim (t-s)^{\delta_1}\varepsilon^{2(1-\delta_3-\delta_1)}s^{\delta_3}\|\mathcal{Q}_\varepsilon f\|_{\mathcal{E}^\gamma\mathcal{C}^\alpha_p}
\end{aligned}
$$

□

The following result is essentially a by-product of the previous proof, but deserves a separate statement due to its' importance.

**Lemma 4.17.** *Consider $\alpha, \beta \in \mathbb{R}$ and $p \in [1,\infty]$ with $\gamma := \alpha - \beta \in [0,2]$. Then uniformly over $\varphi$:*

$$\|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\varphi\|_{\mathcal{C}^\beta_p} \lesssim t^{\frac{\gamma}{2}}\|\varphi\|_{\mathcal{C}^\alpha_p}.$$

*Proof.* The proof follows from Proposition 4.16. Indeed, Equation (34) implies that for $i \lesssim \varepsilon^{-1}$ one has:

$$2^{i\beta}\|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\Delta_i\varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}}2^{i\beta}\|\Delta_i\varphi\|_{\mathcal{C}^\gamma_p} \lesssim t^{\frac{\gamma}{2}}\|\varphi\|_{\mathcal{C}^\alpha_p}.$$

While a slight modification (to $L^p$ spaces) of (35) guarantees that for $i \gtrsim \varepsilon^{-1}$:

$$2^{i\beta}\|(e^{t\mathcal{A}_\varepsilon} - \mathrm{Id})\Delta_i\varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}}2^{i\beta}\varepsilon^{-\gamma}\|\Delta_i\varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}}2^{i\alpha}\|\Delta_j\varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}}\|\varphi\|_{\mathcal{C}^\alpha_p}$$

□

## 5. Semidiscrete Parabolic Anderson Model

5.1. **Commutator estimates & paraproducts.** Let us recall some classical results in harmonic analysis .For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$ set

$$S_i\varphi := \sum_{j=-1}^{i-1}\Delta_j\varphi, \qquad \varphi \varolessthan \psi := \sum_{i\ge -1} S_{i-1}\varphi\Delta_i\psi, \qquad \varphi \odot \psi := \sum_{|i-j|\le 1}\Delta_j\varphi\Delta_i\psi,$$

where the latter sum might not be well defined. Then, an *a priori* ill-posed product of $\varphi$ and $\psi$ can be written as

$$\varphi\cdot\psi = \varphi \varolessthan \psi + \varphi \odot \psi + \varphi \varogreaterthan \psi.$$

The following estimates are classical, see e.g. [3, Lemmata 2.82 and 2.85].

**Lemma 5.1.** *Let $\alpha, \beta \in \mathbb{R}$ and fix $p, q, r \in [1, \infty]$ such that $1/r = 1/p+1/q$. For $\varphi, \psi \in C([0,T]; \mathcal{S}'(\mathbb{T}^d))$*

$$\|\varphi \varolessthan \psi\|_{\mathcal{C}^\alpha_r} \lesssim \|\varphi\|_{L^p} \|\psi\|_{\mathcal{C}^\alpha_q},$$
$$\|\varphi \varolessthan \psi\|_{\mathcal{C}^{\alpha+\beta}_r} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha_q}, \quad \text{if } \beta < 0,$$
$$\|\varphi \odot \psi\|_{\mathcal{C}^{\alpha+\beta}_r} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha_q} \quad \text{if } \alpha+\beta > 0.$$

We define commutators which play an important role in our arguments.

**Definition 5.2.** *For distributions $\varphi, \psi, \sigma \in \mathcal{S}'(\mathbb{T}^d)$ we define the a-priori ill-posed commutators*

$$C^\odot(\varphi, \psi, \sigma) := \varphi \odot (\psi \varolessthan \sigma) - \psi(\varphi \odot \sigma),$$
$$C^\Pi_\varepsilon(\varphi, \psi) := \Pi^2_\varepsilon(\varphi \varolessthan \psi) - \varphi \varolessthan \Pi^2_\varepsilon \psi.$$

*If $\varphi.\psi$ are time-dependent, define additionally*

$$C_{\varepsilon,\lambda}(\varphi, \psi) := (-\mathcal{A}_\varepsilon + \lambda)^{-1}(\varphi \varolessthan \psi) - \varphi \varolessthan (-\mathcal{A}_\varepsilon + \lambda)^{-1}\psi.$$

In the following we prove some results concerning these commutators.

**Lemma 5.3.** *For $\varphi, \psi, \sigma \in \mathcal{S}'(\mathbb{T}^d)$, $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma > 0$ and $p \in [1, \infty]$:*

$$\|C^\odot(\varphi, \psi, \sigma)\|_{\mathcal{C}^{\alpha+\gamma}_p} \lesssim \|\varphi\|_{\mathcal{C}^\alpha} \|\psi\|_{\mathcal{C}^\beta_p} \|\sigma\|_{\mathcal{C}^\gamma}.$$

**Lemma 5.4.** *For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$ and $\alpha \in \mathbb{R}, \beta > 0, p \in [1, \infty]$ it holds for every $\delta \in (0, \beta) \cap (0, 1)$:*

$$\|\mathcal{P}_\varepsilon C^\Pi_\varepsilon(\varphi, \psi)\|_{\mathcal{C}^{\alpha+\delta}_p} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha}, \qquad \|\mathcal{Q}_\varepsilon C^\Pi_\varepsilon(\varphi, \psi)\|_{\mathcal{C}^\alpha_p} \lesssim \varepsilon^\delta \|\varphi\|_{\mathcal{C}^\beta_p} \|\psi\|_{\mathcal{C}^\alpha}.$$

Add reference EBP lecture notes.

*Proof.* Note that for any $i \geq 0$ there exists an annulus $\mathcal{A}$ such that the Fourier transform of

$$\Pi^2_\varepsilon[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi^2_\varepsilon \Delta_i \varphi$$

is contained in $2^i \mathcal{A}$. It is therefore sufficient to show that

$$\big\|\Pi^2_\varepsilon[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi^2_\varepsilon \Delta_i \varphi\big\|_{L^p} \lesssim \varepsilon^\delta \|\varphi\|_{\mathcal{C}^\beta_p} \|\Delta_i \psi\|_{L^\infty}, \tag{36}$$

since this implies the required bound by estimating $\varepsilon^\delta \lesssim 2^{-\delta i}$ for $i$ such that $\mathcal{P}_\varepsilon \Delta_i \neq 0$. To obtain (36), recall the Sobolev-Slobodeckij characterization of fractional spaces 4.4 and observe that $\delta \in (0, 1)$)

$$\begin{aligned}
\|\Pi^2_\varepsilon[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi^2_\varepsilon \Delta_i \varphi\|_\infty &\leq \bigg( \int_{\mathbb{T}^d} \Big| \fint_{B_\varepsilon(x)} [S_{i-1}\varphi(y) - S_{i-1}\varphi(x)] \Delta_i \psi(y) \, \mathrm{d}y \Big|^p \mathrm{d}x \bigg)^{\frac{1}{p}} \\
&\lesssim \varepsilon^\delta \bigg( \int_{\mathbb{T}^d} \Big| \fint_{B_\varepsilon(x)} \frac{[S_{i-1}\varphi(y) - S_{i-1}\varphi(x)]}{|y-x|^\delta} \Delta_i \psi(y) \, \mathrm{d}y \Big|^p \mathrm{d}x \bigg)^{\frac{1}{p}} \\
&\lesssim \varepsilon^\delta \bigg( \int_{\mathbb{T}^d} \int_{B_\varepsilon(x)} \frac{|S_{i-1}\varphi(y) - S_{i-1}\varphi(x)|^p}{|y-x|^{d+\delta p}} \, \mathrm{d}y \, \mathrm{d}x \bigg)^{\frac{1}{p}} \|\Delta_i \psi\|_\infty \\
&\lesssim \varepsilon^\delta \|S_{i-1}\varphi\|_{\mathcal{C}^\beta_p} 2^{-\alpha i} \|\psi\|_{\mathcal{C}^\alpha},
\end{aligned}$$

where the first inequality follows by Jensen's inequality and and we have used the embedding $B^\beta_{p,\infty} \subset B^\delta_{p,p}$. Now the result follows since:

$$\|S_{i-1}\varphi\|_{\mathcal{C}^\beta_p} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p}.$$

This concludes the proof. ∎

**Lemma 5.5.** *For $\varphi, \psi \in \mathcal{S}'(\mathbb{T}^d)$ and $\alpha \in (0,2), \beta \in \mathbb{R}$ and $p \in [1,\infty]$*

$$\|\mathcal{P}_\varepsilon C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^{\beta+2}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^{\beta}} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}$$

*Proof.* By the elliptic Schauder estimate of Lemma 4.15, it is sufficient to prove that

$$\|(-\mathcal{A}_\varepsilon+\lambda)C_{\varepsilon,\lambda}(\varphi,\psi)\|_{\mathcal{C}_p^\beta} \lesssim \|\varphi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}.$$

In turn to obtain this bound, taking into account the support of the Fourier transform of the quantities below,it suffices to estimate for $i \geq 0$

$$\|S_{i-1}\varphi\Delta_i\psi - (-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi]\|_{L^p} \lesssim 2^{-i\beta}\|\phi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta}. \tag{37}$$

Let $B_\varepsilon(\varphi,\psi)$ be defined as

$$B_\varepsilon(\varphi,\psi)(x) = \varepsilon^{-2}⨍_{B_\varepsilon(x)} \mathrm{d}y ⨍_{B_\varepsilon(y)} \mathrm{d}z\ (\varphi(z)-\varphi(x))(\psi(z)-\psi(x)).$$

Then $\mathcal{A}_\varepsilon$ can be decomposed as

$$\mathcal{A}_\varepsilon(\varphi\cdot\psi) = \mathcal{A}_\varepsilon(\varphi)\cdot\psi + \varphi\cdot\mathcal{A}_\varepsilon(\psi) + B_\varepsilon(\varphi,\psi),$$

Hence proving Equation (37) reduces to finding a bound for

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi](-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} + \|B_\varepsilon(S_{i-1}\varphi, (-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi)\|_{L^p}.$$

Starting with the first term, one has:

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi](-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} \lesssim \|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p}\|(-\mathcal{A}_\varepsilon+\lambda)^{-1}[\Delta_i\psi]\|_{L^\infty}.$$

If $2^{-i} \gtrsim \varepsilon$, since $\alpha < 2$, one can estimate via Lemma 4.11

$$\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p} \leq \sum_{j=-1}^{i-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p} \lesssim \sum_{j=-1}^{i-1} 2^{j(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha} \lesssim 2^{i(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha}.$$

If $2^{-i} \lesssim \varepsilon$ choose $i(\varepsilon)$ such that $2^{-i(\varepsilon)} \sim \varepsilon$. Then following the previous calculations and using that $\alpha > 0$:

$$\begin{aligned}\|(-\mathcal{A}_\varepsilon+\lambda)[S_{i-1}\varphi]\|_{L^p} &\leq \sum_{j=-1}^{i(\varepsilon)-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p} + \sum_{j=i(\varepsilon)}^{i-1}\|(-\mathcal{A}_\varepsilon+\lambda)[\Delta_j\varphi]\|_{L^p}\\ &\lesssim \varepsilon^{-(2-\alpha)}\|\varphi\|_{\mathcal{C}_p^\alpha}.\end{aligned}$$

By Proposition 4.16

$$\|(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi\| \lesssim \Big(2^{-2i}1_{\{2^{-i}\gtrsim\varepsilon\}} + \varepsilon^2 1_{\{2^{-i}\lesssim\varepsilon\}}\Big)2^{-\beta i}\|\psi\|_{\mathcal{C}^\beta},$$

which provides a bound of the required order for (37). Finally, we have to bound the term containing $B_\varepsilon$. If $2^{-i} \gtrsim \varepsilon$

$$\begin{aligned}&\|\nabla S_{i-1}\varphi\|_{L^p}\|\nabla(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi\|_{L^\infty}\\ &\qquad\lesssim 2^i\|S_{i-1}\varphi\|_{L^p}2^{-(1+\beta)i}\|\psi\|_{\mathcal{C}^\beta}\\ &\qquad\lesssim 2^{-\beta i}\|\varphi\|_{\mathcal{C}_p^\alpha}\|\psi\|_{\mathcal{C}^\beta},\end{aligned}$$

whereas if $2^{-i} \lesssim \varepsilon$

$$\begin{aligned}\|B_\varepsilon(S_{i-1}\varphi, (-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi)\|_{L^p} &\lesssim \varepsilon^{-2}\|S_{i-1}\varphi\|_{L^p}\|(-\mathcal{A}_\varepsilon+\lambda)^{-1}\Delta_i\psi\|_{L^\infty}\\ &\lesssim 2^{-\beta i}\|\varphi\|_{L^p}\|\psi\|_{\mathcal{C}^\beta}.\end{aligned}$$

This bound is again of the correct order for (37) This concludes the proof.

maybe explain at the beginning what it means with $\sim \varepsilon$ in the indices

□

5.2. **The semidiscrete parabolic Anderson model.** A simple corollary of the Proposition 6.1 is the following (which can be regarded as a definition of the norm $|||\cdot|||_\varepsilon$). The presence of the factor $\frac{\kappa}{2}$ below is simply for convenience in the proof of Proposition 5.7

**Lemma 5.6.** *Fix any $\kappa > 0, \lambda > 1$. Let $\xi_\varepsilon$ satisfy Assumption 1.4 and up to changing probability space consider $\xi = \bar{\xi}$ as described in Proposition 6.1. Recall that*

$$X_{\varepsilon,\lambda}(\omega) = (-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon(\omega), \qquad \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}(\omega) = \xi_\varepsilon(\omega) \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda}(\omega) - c_\varepsilon,$$

*where $c_\varepsilon$ is defined in Proposition 6.1 and satisfies*

$$|c_\varepsilon| \lesssim \log \frac{1}{\varepsilon}.$$

*Define for $\omega \in \Omega$:*

$$\begin{aligned} |||\xi_\varepsilon(\omega)|||_\varepsilon = \sup_{\zeta \in [0,1]} \varepsilon^{\frac{d}{2}\zeta} \Big\{ &\|\xi_\varepsilon(\omega)\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)-\frac{\kappa}{2}}} + \|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}(\omega)\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)+2-\frac{\kappa}{2}}} \\ &+ \varepsilon^{-2} \|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}(\omega)\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)-\frac{\kappa}{2}}} \Big\} + \|\xi_\varepsilon(\omega) \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}(\omega)\|_{\mathcal{C}^{-\frac{\kappa}{2}}} 1_{\{d=2\}} \\ &+ \varepsilon^{\frac{d}{2}} \|\xi_\varepsilon\|_{L^\infty} + \varepsilon^{2-\frac{d}{2}} \|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty}. \end{aligned} \tag{38}$$

*Then for almost all $\omega \in \Omega$ the following bounds hold:*

$$\sup_{\varepsilon \in (0,1)} |||\xi_\varepsilon(\omega)|||_\varepsilon < \infty$$

*and there exist limiting distributions such that: $\xi_\varepsilon(\omega) \to \xi$ in $\mathcal{C}^{-\frac{d}{2}-\frac{\kappa}{2}}$, $\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}(\omega) \to (-\Delta+\lambda)^{-1}\xi$, in $\mathcal{C}^{2-\frac{d}{2}-\frac{\kappa}{2}}$, and $\xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}(\omega) \to \xi \diamond X_\lambda$ in $\mathcal{C}^{-\frac{\kappa}{2}}$.*

*Proof.* In view of Proposition 6.1, for fixed $\zeta = 0$ or $\zeta = 1$ the right hand-side of (38) is bounded uniformly in $\varepsilon$. The result uniformly over $\zeta$ follows by the following interpolation inequality for $\vartheta \in [0,1]$ and $\alpha, \beta \in \mathbb{R}$:

$$\|\varphi\|_{\mathcal{C}^{\vartheta\alpha+(1-\vartheta)\beta}} \leq \|\varphi\|^{\vartheta}_{\mathcal{C}^\alpha} \|\varphi\|^{1-\vartheta}_{\mathcal{C}^\beta}.$$

This concludes the proof. □

**Proposition 5.7.** *Fix $\kappa > 0$ and $\omega \in \Omega$ and let*

$$\mathcal{H}(\omega) = \Delta + \xi(\omega) - \infty 1_{\{d=2\}}$$

*be the Anderson Hamiltonian associated to $\xi(\omega)$, as constructed in [2]. The Hamiltonian has discrete spectrum given by pairs of eigenvalues and eigenfunctions $\{(\lambda_k(\omega), e_k(\omega))\}_{k\in\mathbb{N}}$ such that:*

$$\lambda_0(\omega) > \lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots, \qquad \lim_{k\to\infty} \lambda_k(\omega) = -\infty, \qquad e_0(\omega, x) > 0, \forall x \in \mathbb{T}^d.$$

*Moreover, for every $k \in \mathbb{N}$, $e_k(\omega) \in \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$, and finite linear combinations of $\{e_k(\omega)\}_{k\in\mathbb{N}}$ form a dense set in $\mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d)$.*

*Proof.* probably a point should be made about pathwise The Anderson Hamiltonian $\mathcal{H}$ has been constructed by [2] as an unbounded, selfadjoint operator on $L^2$, that is

$$\mathcal{H} \colon \mathcal{D}(\mathcal{H}) \subset L^2 \to L^2.$$

In particular, [2, Proposition 4.13] implies that the operator $\mathcal{H}$ admits compact resolvents. Hence the spectrum of $\mathcal{H}$ is discrete and the eigenvalues converge to $-\infty$. By a classical result, see [24, Theorem 3.3], the semigroup generated by $\mathcal{H}$, denoted by $e^{t\mathcal{H}}$, is compact. Moreover, as a consequence of strong maximum principle for singular stochastic PDEs applied to Parabolic Anderson Model, see [6, Theorem 5.1 and Remark 5.2], the semigroup $e^{t\mathcal{H}}$ is strictly positive, that is for any non-zero function $f$ such that $\forall x \in \mathbb{T}^d e^{t\mathcal{H}} f(x) > 0$. Therefore since $e^{t\mathcal{H}}$ is a compact, strictly positive operator, the Krein-Rutman Theorem implies that the first eigenfunction is strictly positive, with a positive egenvalue of multiplicity one.

To conclude the analysis of spectral properties of $\mathcal{H}$ it remains to show the regularity of the the eigenfunctions. Since the arguments are similiar to both those present in [2] and to proof of Proposition 5.8, a discussion of this point is omitted. □

**Proposition 5.8.** *For every $k \in \mathbb{N}$ there exists a $\varepsilon_0(\omega, k) \in (0, 1/2)$ such that for every $\varepsilon \leq \varepsilon_0(\omega, k)$ there exists a pair of eigenvalue and associated eigenfunction $(\lambda_k^\varepsilon(\omega), e_k^\varepsilon)(\omega)$ for the operator*

$$\mathcal{H}_\varepsilon(\omega) := \mathcal{A}_\varepsilon + (\xi_\varepsilon(\omega) - c_\varepsilon)\Pi_\varepsilon^2,$$

*such that*

$$\lim_{\varepsilon \to 0} \lambda_k^\varepsilon(\omega) = \lambda_k(\omega), \qquad \lim_{\varepsilon \to 0} \Pi_\varepsilon e_k^\varepsilon(\omega) = e_k(\omega) \quad \text{in} \quad \mathcal{C}^{2-\frac{d}{2}-\kappa}(\mathbb{T}^d).$$

*Proof.* We restrict our attention to dimension $d = 2$. In dimension $d = 1$ the proof follows by similar, but simpler, arguments. Since the proof is based on a pathwise analysis, that is, the $\omega\Omega$ is fixed, the dependence on $\omega$ is omitted.

For $\lambda \in \mathbb{R}$ define

$$\mathcal{H}_{\varepsilon,\lambda} \colon L^2 \to L^2, \qquad \mathcal{H}_{\varepsilon,\lambda}\psi = (\mathcal{A}_\varepsilon + (\xi_\varepsilon - c_\varepsilon)\Pi^2 - \lambda)\psi.$$

The proof is divided into four steps. In the first three steps, we prove that there exists a $\bar{\lambda} > 0$ such that for all $\lambda \geq \bar{\lambda}$ and $\varepsilon \in (0, 1/2)$ the operator $\mathcal{H}_{\varepsilon,\lambda}$ is invertible and

$$\lim_{\varepsilon \to 0} \|\mathcal{H}_{\varepsilon,\lambda}^{-1} - (\mathcal{H} - \lambda)^{-1}\|_{B(L^2,L^2)} = 0 \tag{39}$$

where $B(X, Y)$ is the space of bounded linear operators between two Banach spaces $X, Y$ with the standard operator norm. By the continuity of the spectrum, see [19, Chapter 4, Theorem 3.16], and (39), it follows that for any $k \in \mathbb{N}$ there exists a $\varepsilon_0(k)$ and eigenvalues and associated an associated eigenfunction $(\lambda_k^\varepsilon, e_k^\varepsilon) \in \mathbb{R} \times L^2$ such that

$$\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda_k, \qquad \lim_{\varepsilon \to 0} e_k^\varepsilon = e_k \quad \text{in} \quad L^2.$$

The strategy of the proof is somewhat similiar to [2] and based on a fixed point argument. The precise setup for the fixed point argument is discussed in Step 1. The estimates which guarantees its' validity, including some regularity estimates, are discussed in Step 2, followed by closure of the argument in Step 3. In the fourth step those estimate are used to show that $\Pi_\varepsilon e_k^\varepsilon \to e_k \in \mathcal{C}^{1-\kappa}$.

*Step 1.* Fix $p \in [1, \infty]$ as well as $\varphi \in \mathcal{C}_p^{-1+2\kappa}$. In dimension $d = 1$, solving the resolvent equation $\mathcal{H}_{\varepsilon,\lambda}\psi = \varphi$ is equivalent to solving the fixed point problem

$$\psi = M_{\varphi,\lambda}(\psi) := (-\mathcal{A}_\varepsilon + \lambda)^{-1}[(\xi_\varepsilon - c_\varepsilon)\Pi_\varepsilon^2\psi - \varphi]. \tag{40}$$

In dimension $d = 2$, $M_{\varphi,\lambda}(\psi)$, treated in a classical way, is insufficient. To find the fixed point we look for a *paracontrolled* solution. Consider a space $\mathcal{D}_\varepsilon \subseteq L^p(\mathbb{T}^d) \times L^p(\mathbb{T}^d)$ which, for a pair $(\psi', \psi^\sharp)$ is characterized by the norm

$$\|(\psi', \psi^\sharp)\|_{\mathcal{D}_\varepsilon} := \|\psi'\|_{\mathcal{C}_p^{1-\kappa}} + \|\mathcal{P}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{1+\kappa}} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon\psi^\sharp\|_{\mathcal{C}_p^{-1+2\kappa}}.$$

A function $\psi$ is associated to a pair $(\psi', \psi^\sharp)$ by

$$\psi = \psi' \varolessthan [(-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon] + \psi^\sharp.$$

With an abuse of notation, we identify the pair $(\psi', \psi^\sharp)$ with the function $\psi$ and write $\|\psi\|_{\mathcal{D}_\varepsilon} = \|(\psi', \psi^\sharp)\|_{\mathcal{D}_\varepsilon}$. Define a map (note the presence of $\psi'$) $\overline{M}_{\varphi,\lambda} \colon \mathcal{D}_\varepsilon \to L^p$ as

$$\overline{M}_{\varphi,\lambda}(\psi) := (-\mathcal{A}_\varepsilon + \lambda)^{-1}[\xi_\varepsilon\Pi_\varepsilon^2\psi - c_\varepsilon\psi' - \varphi].$$

The map $\overline{M}_{\varphi,\lambda}$ can be extended to a map from $\mathcal{D}_\varepsilon$ into itself by

$$\mathcal{M}_{\varphi,\lambda}(\psi) = (M'_{\varphi,\lambda}(\psi),\ M^\sharp_{\varphi,\lambda}(\psi)) := (\Pi_\varepsilon^2\psi,\ \overline{M}_{\varphi,\lambda}(\psi) - (\Pi_\varepsilon^2\psi) \varolessthan [(-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon]) \in \mathcal{D}_\varepsilon,$$

The fixed point of $\mathcal{M}_\varphi$ solves (40) as well, since the fixed point satisfies

$$\psi' = \Pi_\varepsilon^2\psi.$$

*Step 2.* In the course of the proof we repeatedly make use of the elliptic Schauder estimates of Lemma 4.15 and the paraproduct estimates of Lemma 5.1, without stating them explicitly every time. The aim is to control

$$\|\mathcal{M}_{\varphi,\lambda}(\psi)\|_{\mathcal{D}_\varepsilon} = \|\Pi^2_\varepsilon \psi\|_{\mathcal{C}^{1-\kappa}_p} + \|\mathcal{P}_\varepsilon \, M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathcal{C}^{1+\kappa}_p} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon \, M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathcal{C}^{-1+2\kappa}_p}.$$

Via Lemma 4.10:

$$\begin{aligned}\|\Pi^2_\varepsilon \psi\|_{\mathcal{C}^{1-\kappa}_p} &\lesssim \|\Pi^2_\varepsilon\big[\psi' \varolessthan X_{\varepsilon,\lambda}\big]\|_{\mathcal{C}^{1-\kappa}_p} + \|\mathcal{P}_\varepsilon \psi^\sharp\|_{\mathcal{C}^{1+\kappa}_p} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon \psi^\sharp\|_{\mathcal{C}^{-1+2\kappa}_p} \\ &\lesssim \lambda^{-\frac{\kappa}{4}}\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}(\|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{1-\frac{\kappa}{2}}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}) \\ &\qquad + \|\mathcal{P}_\varepsilon \psi^\sharp\|_{\mathcal{C}^{1+\kappa}_p} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon \psi^\sharp\|_{\mathcal{C}^{-1+2\kappa}_p}.\end{aligned} \tag{41}$$

To tackle the norms $M^\sharp$, first rewrite as

$$\begin{aligned}M^\sharp_\varphi(\psi) = (-\mathcal{A}_\varepsilon+\lambda)^{-1}\Big\{ -\varphi + [\xi_\varepsilon \odot \Pi^2_\varepsilon \psi^\sharp] + \big\{\xi_\varepsilon \odot [\Pi^2_\varepsilon(\psi' \varolessthan X_{\varepsilon,\lambda})] - c_\varepsilon \psi'\big\} \\ + \xi_\varepsilon \varolessthan \Pi^2_\varepsilon \psi + C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi, \xi_\varepsilon)\Big\},\end{aligned}$$

where $C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon)$ is the commutator

$$C_{\varepsilon,\lambda}(\Pi^2_\varepsilon\psi,\xi_\varepsilon) = (-\mathcal{A}_\varepsilon+\lambda)^{-1}[(\Pi^2_\varepsilon\psi)\varolessthan \xi_\varepsilon] - [(\Pi^2_\varepsilon\psi)\varolessthan(-\mathcal{A}_\varepsilon+\lambda)^{-1}(\xi_\varepsilon)].$$

Combining the Schauder estimates with the smoothing properties of $\Pi_\varepsilon$ and the paraproduct estimates one finds that

$$\begin{aligned}\lambda^{\frac{\kappa}{2}}\big(\|\mathcal{P}_\varepsilon M^\sharp_\varphi(\psi)\|_{\mathcal{C}^{1+\kappa}_p} + \varepsilon^{-2+\kappa}\|\mathcal{Q}_\varepsilon M^\sharp_\varphi(\psi)\|_{\mathcal{C}^{-1+2\kappa}_p}\big) \lesssim & \|\varphi\|_{\mathcal{C}^{-1+2\kappa}_p} + \|\Pi^2_\varepsilon\psi^\sharp\|_{\mathcal{C}^{1+\kappa}_p}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}} \\ &+ \|\xi_\varepsilon \odot [\Pi^2_\varepsilon(\psi'\varolessthan X_{\varepsilon,\lambda})] - c_\varepsilon\psi'\|_{\mathcal{C}^{-1+2\kappa}_p} \\ &+ \|C_{\varepsilon,\lambda}(\psi,\xi_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}.\end{aligned}$$

To treat $\|\xi_\varepsilon \odot [\Pi^2_\varepsilon(\psi'\varolessthan X_{\varepsilon,\lambda}) - c_\varepsilon\psi']\|_{\mathcal{C}^{-1+2\kappa}_p}$, we introduce (cf. 5.2) notations

$$C^\Pi_\varepsilon(f,g) = \Pi^2_\varepsilon(f\varolessthan g) - f\varolessthan \Pi^2_\varepsilon g, \qquad C^\odot(f,g,h) = f\odot(g\varolessthan h) - g(f\odot h).$$

By via Lemma 5.4

$$\begin{aligned}\|\xi_\varepsilon \odot C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p} &\le \|\xi_\varepsilon\odot C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^\kappa_p} \\ &\lesssim \|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}\|\mathcal{P}_\varepsilon C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{1+\kappa}_p} \\ &\quad + \|\xi_\varepsilon\|_{\mathcal{C}^{-1+\kappa}}\|\mathcal{Q}_\varepsilon C^\Pi_\varepsilon(\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{1-\frac{\kappa}{2}}_p} \\ &\lesssim \|\psi'\|_{\mathcal{C}^{1-\kappa}_p}|||\xi_\varepsilon|||^2_\varepsilon.\end{aligned}$$

We deduce that

$$\begin{aligned}\|\xi_\varepsilon\odot[\psi'\varolessthan(\Pi^2_\varepsilon X_{\varepsilon,\lambda})] - c_\varepsilon\psi'\|_{\mathcal{C}^{-1+2\kappa}_p} \lesssim & \|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p} \\ &+ \|\psi'(\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda} - c_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p}.\end{aligned}$$

By Lemma 5.3

$$\begin{aligned}\|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-1+2\kappa}_p} &\le \|C^\odot(\xi_\varepsilon,\psi',X_{\varepsilon,\lambda})\|_{\mathcal{C}^{-2\kappa}_p} \\ &\lesssim \|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}}\|\psi'\|_{\mathcal{C}^{1-\kappa}_p}\|\Pi^2_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{1-\frac{\kappa}{2}}} \lesssim \|\psi'\|_{\mathcal{C}^{1-\kappa}_p}|||\xi_\varepsilon|||^2_\varepsilon.\end{aligned}$$

Similarly:

$$\|\psi'(\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}-c_\varepsilon)\|_{\mathcal{C}^{-1+2\kappa}_p} \lesssim \|\psi'\|_{\mathcal{C}^{1-\kappa}_p}\|\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}-c_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}} \le \|\psi'\|_{\mathcal{C}^{1-\kappa}_p}|||\xi_\varepsilon|||_\varepsilon.$$

The estimate for $C_{\varepsilon,\lambda}(\Pi_\varepsilon^2\psi,\xi_\varepsilon)$ follows from Lemma 5.5, by noticing that

$$\begin{aligned}\|C_{\varepsilon,\lambda}(\Pi_\varepsilon^2\psi,\xi_\varepsilon)\|_{\mathcal{C}_p^{-1+2\kappa}} &\le \|\mathcal{P}_\varepsilon C_{\varepsilon,\lambda}(\Pi_\varepsilon^2\psi,\xi_\varepsilon)\|_{\mathcal{C}_p^{-1+2\kappa}} + \|\mathcal{Q}_\varepsilon C_{\varepsilon,\lambda}(\Pi_\varepsilon^2\psi,\xi_\varepsilon)\|_{\mathcal{C}_p^{-1+2\kappa}}\\ &\lesssim \|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}}\|\xi_\varepsilon\|_{\mathcal{C}^{-1-\frac{\kappa}{2}}} + \varepsilon^2\|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}}\|\xi_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}}\\ &\lesssim \|\psi\|_{\mathcal{D}_\varepsilon}|||\xi_\varepsilon|||_\varepsilon.\end{aligned}$$

*Step 3.* Estimates from Step 2. combined with linearity guarantee that that for $n\in\mathbb{N}, n\ge 2$ there exists a $C>0$ such that

$$\begin{aligned}\|\mathcal{M}_\varphi(\psi)\|_{\mathcal{D}_\varepsilon} &\le C\Big[\|\varphi\|_{B_{p,q}^{-1+2\kappa}} + \|\psi\|_{\mathcal{D}_\varepsilon}(1+|||\xi_\varepsilon|||_\varepsilon)^2\Big]\\ \|\big[\mathcal{M}_\varphi(\psi)-\mathcal{M}_\varphi(\tilde\psi)\big]^n\|_{\mathcal{D}_\varepsilon} &\le C\Big[\lambda^{-\frac{\kappa}{4}}\|\psi-\tilde\psi\|_{\mathcal{D}_\varepsilon}(1+|||\xi_\varepsilon|||_\varepsilon)^2\Big].\end{aligned}$$

Note that we require $n\ge 2$, since in (41) we do not have a small factor in front of the rest term with $\psi^\sharp$. In particular, there exists a $\bar\lambda(\sup_\varepsilon|||\xi_\varepsilon|||_\varepsilon)$ such that for $\lambda>\bar\lambda$ the map $\mathcal{M}_\varphi$ admits a unique fixed point, which we denote by $\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi$. Moreover, by Banach fixed point Theorem

$$\|\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi\|_{\mathcal{D}_\varepsilon} \lesssim \|\mathcal{M}_\varphi^2(0)\|_{\mathcal{D}_\varepsilon} \lesssim \|\varphi\|_{\mathcal{C}_p^{-1+2\kappa}}(1+|||\xi_\varepsilon|||_\varepsilon)^2,$$

implying that $\mathcal{H}_{\varepsilon,\lambda}^{-1}\in B(\mathcal{C}_p^{-1+2\kappa},\mathcal{D}_\varepsilon)$, with the norm bounded uniformly in $\varepsilon$. Simlar, but less involved calculations lead to a construction of the resolvent $\mathcal{H}_\lambda^{-1}=(\mathcal{H}-\lambda)^{-1}$ in the continuum for $\lambda\ge\bar\lambda$. This is a bounded operator $\mathcal{H}_\lambda^{-1}\in B(\mathcal{C}_p^{-1+2\kappa},\mathcal{D}_0)$, where the latter is the Banach space defined by the norm (for $\psi=\psi'\otimes(-\Delta+\lambda)^{-1}\xi+\psi^\sharp$):

$$\|\psi\|_{\mathcal{D}_0} = \|\psi'\|_{\mathcal{C}_p^{1-\kappa}} + \|\psi^\sharp\|_{\mathcal{C}_p^{1+\kappa}}.$$

Similarly, by linearity and computations similar to those in Step 2.,

$$\lim_{\varepsilon\to 0}\sup_{\|\varphi\|_{\mathcal{C}_p^{-1+2\kappa}}\le 1}\Big\|(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)'-(\mathcal{H}_\lambda^{-1}\varphi)'\Big\|_{\mathcal{C}_p^{1-\kappa}} + \Big\|\mathcal{P}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp-(\mathcal{H}_\lambda^{-1}\varphi)^\sharp\Big\|_{\mathcal{C}_p^{1+\kappa}} = 0. \tag{42}$$

To show (39) it is now sufficient to show that $\mathcal{D}_\varepsilon\hookrightarrow L^2$. For that purpose a better regularity of $\psi^\sharp$ is required, which can be deduced by using g $\varphi\in L^p$, namely

$$\lambda^{\frac{\kappa}{2}}\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon M_\varphi^\sharp(\psi)\|_{L^p} \lesssim \varepsilon^{3\kappa}\Big\{\|\xi_\varepsilon\Pi_\varepsilon^2\psi - c_\varepsilon\psi'-\varphi\|_{L^p} + \|(\Pi_\varepsilon^2\psi)\otimes X_{\varepsilon,\lambda}\|_{L^p}\Big\}$$

and since $c_\varepsilon\lesssim\log\frac{1}{\varepsilon}$ (see Proposition 6.1)

$$\|\xi_\varepsilon\Pi_\varepsilon^2\psi-c_\varepsilon\psi'-\varphi\|_{L^p} \lesssim \|\xi_\varepsilon\|_{\mathcal{C}^{-1+2\kappa}}\|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}} + \varepsilon^{-\kappa}\|\psi'\|_{L^p} + \|\varphi\|_{L^p}.$$

Similarly, since

$$\|f\otimes g\|_{L^p} \le \|fg\|_{L^p} + \|g\otimes f\|_{L^p} + \|f\odot g\|_{L^p} \lesssim \|f\|_{\mathcal{C}_p^\kappa}\|g\|_{L^\infty}.$$

one has

$$\|(\Pi_\varepsilon^2\psi)\otimes X_{\varepsilon,\lambda}\|_{L^p} \lesssim \|\Pi_\varepsilon^2\psi\|_{\mathcal{C}_p^{1-\kappa}}\|X_{\varepsilon,\lambda}\|_{L^\infty}.$$

Therefore the $L^\infty$ bound on $\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}$ leads to

$$\lambda^{\frac{\kappa}{2}}\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon M_\varphi^\sharp(\psi)\|_{L^p} \lesssim \|\varphi\|_{L^p} + \|\psi\|_{\mathcal{D}_\varepsilon}(1+|||\xi_\varepsilon|||_\varepsilon). \tag{43}$$

In particular, the regularity of the resolvent map $\mathcal{H}_{\varepsilon,\lambda}^{-1}$ is enhanced by

$$\varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp\|_{L^p} \lesssim \|\varphi\|_{L^p}.$$

This leads to embedding $\mathcal{D}_\varepsilon\hookrightarrow L^2$ which justifies

$$\begin{aligned}\|\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi-\mathcal{H}_\lambda^{-1}\varphi\|_{L^2} \le&\|(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)'\otimes X_{\varepsilon,\lambda}-(\mathcal{H}_\lambda^{-1}\varphi)'\otimes(-\Delta+\lambda)^{-1}\xi\|_{L^2}\\ &+\|\mathcal{P}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp-(\mathcal{H}_\lambda^{-1}\varphi)^\sharp\|_{L^2} + \|\mathcal{Q}_\varepsilon(\mathcal{H}_{\varepsilon,\lambda}^{-1}\varphi)^\sharp\|_{L^2}.\end{aligned}$$

Letting $\varepsilon$ to 0 (39) follows from (42), (43) and the stochastic bounds.

*Step 5.* It remains to show that

$$\Pi_\varepsilon e_k^\varepsilon \to e_k \quad \text{in} \quad \mathcal{C}^{1-\kappa}.$$

Since due to compact dependence of $C_p^\alpha$ on $\alpha$, $\kappa$ can be choosen in arbitrary way, and since the convergence already holds in the sense of distributions, it is sufficient to show that

$$\sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}^{1-\kappa}} < \infty.$$

Observe that we already have a uniform bound in $L^2$:

$$\sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{L^2} < \infty.$$

Now, choose $\lambda > \bar{\lambda}$ (and hence $\lambda > \sup_\varepsilon \lambda_k^\varepsilon$), then one can rewrite:

$$e_k^\varepsilon = (\lambda_k^\varepsilon - \lambda)\mathcal{H}_{\varepsilon,\lambda}^{-1} e_k^\varepsilon.$$

So that

$$e_k^\varepsilon = (e_k^\varepsilon)' \otimes X_{\varepsilon,\lambda} + (e_k^\varepsilon)^\sharp,$$

and by all the bounds in the proof of the previous step:

$$\sup_\varepsilon \left\{ \|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon} + \varepsilon^{-2+4\kappa}\|\mathcal{Q}_\varepsilon (e_k^\varepsilon)^\sharp\|_{L^2} \right\} < \infty,$$

where we use the space $\mathcal{D}_\varepsilon$ in the case $p = 2, q = \infty$. Hence, applying Lemma 4.10, one obtains:

$$\begin{aligned}\|\Pi_\varepsilon e_k^\varepsilon\|_{B^{1-\kappa}_{2,\infty}} &\lesssim \|\Pi_\varepsilon\big[(e_k^\varepsilon)' \otimes X_{\varepsilon,\lambda}\big]\|_{B^{1-\kappa}_{2,\infty}} + \|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon} + \varepsilon^{-1}\|\mathcal{Q}_\varepsilon(e_k^\varepsilon)^\sharp\|_{L^2}\Big\} \\ &\lesssim \varepsilon^{-1}\|(e_k^\varepsilon)'\|_{B^{1-\kappa}_{2,\infty}}\|X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{\kappa}{2}}} + \|e_k^\varepsilon\|_{\mathcal{D}_\varepsilon} + \varepsilon^{-1}\|\mathcal{Q}_\varepsilon(e_k^\varepsilon)^\sharp\|_{L^2}\Big\}.\end{aligned}$$

Using the bounds on the noise terms, as well as the uniform bound we already established one thus has that by Besov space embeddings:

$$\sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}^{\frac{1}{2}-\kappa}} \lesssim \sup_\varepsilon \|\Pi_\varepsilon e_k^\varepsilon\|_{\mathcal{C}_2^{1-\kappa}} < \infty.$$

Iterating the entire procedure again in $L^\infty$ instead of $L^2$, one obtains the required uniform bound. □

**Lemma 5.9.** *Fix $\omega \in \Omega$. Consider the Anderson Hamiltonian $\mathcal{H}(\omega)$ as in the previous Proposition. Define the domain:*

$$\mathcal{D}_\omega = \{\textit{Finite linear combinations of the eigenfunctions } \{e_k(\omega)\}_{k\in\mathbb{N}}\}.$$

*Such domain is dense in $C(\mathbb{T}^d)$. Moreover, for $\varphi \in C^\infty$ and $\zeta < 1$ there exists a sequence $\varphi^k \in \mathcal{D}$ with $\lim_{k\to\infty} \varphi^k = \varphi$ in $\mathcal{C}^\zeta$.*

*Proof.* Since $\omega \in \Omega$ is fixed, we avoid writing it to lighten the notation. Since statement regarding the approximation of $\varphi$ in $\mathcal{C}^\zeta$ implies density in $C(\mathbb{T}^d)$ we restrict to proving the latter. First, we require some better understanding of the parabolic Anderson semigroup.

*Step 1.* Consider the operator $\mathcal{H}$ constructed in Proposition **??**: Taking an exponential one can construct the semigroup:

$$e^{t\mathcal{H}} \colon L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d).$$

This semigroup inherits some of the regularizing properties of the heat semigroup, namely, for $T > 0$ and $p \in [1, \infty]$ it can be extended so that:

$$\sup_{0<t\leq T} t^\gamma \|e^{t\mathcal{H}}\varphi\|_{\mathcal{C}_p^\alpha} \lesssim \|\varphi\|_{\mathcal{C}_p^\beta}, \tag{44}$$

for $\alpha$ and $\beta$ satisfying:

$$\gamma > \frac{\alpha-\beta}{2}, \qquad \beta + 2 > \frac{d}{2}, \qquad \alpha < 2 - \frac{d}{2}, \qquad \alpha > \beta.$$

The first constraint is essentially identical to the one appearing in Proposition 4.16, the second one guarantess that $e^{t\Delta}\varphi\cdot\xi$ is a well-defined product of distributions, while the third constraint is due to the fact that $\int_0^t e^{(t-s)\Delta}\xi\,\mathrm{d}s$ has always worse regularity than $2-\frac{d}{2}$. We will not prove these results: instead we refer to [25, Proposition 3.1] and the reference therein (the cited proposition is set on the entire space, with the added complication of weights at infinity. Such case contains the current setting by extending the noise periodically). The same results guarantee that in the case $\beta > 2-\frac{d}{2}$ and for $\zeta < 2-\frac{d}{2}$ one has:

$$\sup_{0\le t\le T}\|e^{t\mathcal{H}}\varphi\|_{\mathcal{C}^\zeta_p} \lesssim \|\varphi\|_{\mathcal{C}^\beta_p}. \tag{45}$$

*Step 2.* Now we prove the statement regarding the approximability of $\varphi$. For any $\varphi\in C^\infty$ and $\zeta = 1-\kappa < 1$ (for some $\kappa > 0$) one has:

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t e^{s\mathcal{H}}\varphi\,\mathrm{d}s = \varphi \quad \text{in} \quad \mathcal{C}^\zeta.$$

This can be seen as follows: Equation (45) implies that

$$\sup_{0\le t\le T}\left\|\frac{1}{t}\int_0^t e^{s\mathcal{H}}\varphi\,\mathrm{d}s\right\|_{\mathcal{C}^{\zeta'}} < \infty,$$

for $\zeta < \zeta' < 2-\frac{d}{2}$. This guarantees compactness in $\mathcal{C}^\zeta$. Projecting on the eigenfunctions $e_k$ one sees that any limit point is necessarily $\varphi$. Hence fix any $\varepsilon > 0$ and choose $t(\varepsilon)$ such that

$$\left\|\frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)} e^{s\mathcal{H}}\varphi\,\mathrm{d}s - \varphi\right\|_{\mathcal{C}^\zeta} < \frac{\varepsilon}{2}.$$

Define $\Pi_{\le N}\varphi = \sum_{k=0}^N \langle\varphi, e_k\rangle e_k$. Since the projection commutes with the operator, the proof is complete if we can show that there exists an $N(\varepsilon)$ such that:

$$\left\|\frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)} e^{s\mathcal{H}}(\Pi_{\le N(\varepsilon)}\varphi - \varphi)\,\mathrm{d}s\right\|_{\mathcal{C}^\zeta} \le \frac{\varepsilon}{2}.$$

Here we use Equation (44) to bound for general $\psi\in L^2$:

$$\begin{aligned}\left\|\frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)} e^{s\mathcal{H}}\psi\,\mathrm{d}s\right\|_{\mathcal{C}^\zeta} &\lesssim \frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)}\left(\frac{s}{2}\right)^{-\left(\frac12-\frac{\kappa}{4}\right)}\|e^{\frac{s}{2}\mathcal{H}}\psi\|_{\mathcal{C}^{-\frac{\kappa}{2}}}\,\mathrm{d}s\\ &\lesssim \frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)}\left(\frac{s}{2}\right)^{-\left(\frac12-\frac{\kappa}{4}\right)}\|e^{\frac{s}{2}\mathcal{H}}\psi\|_{\mathcal{C}_2^{\frac{d}{2}-\frac{\kappa}{2}}}\,\mathrm{d}s\\ &\lesssim \frac{1}{t(\varepsilon)}\int_0^{t(\varepsilon)} s^{-1+\frac{\kappa}{4}+\frac{\kappa}{8}}\,\mathrm{d}s\|\psi\|_{L^2} \lesssim t(\varepsilon)^{-1+\frac{3\kappa}{8}}\|\psi\|_{L^2},\end{aligned}$$

where we additionaly applied Besov embeddings. Choosing $N(\varepsilon)$ such that $\|\Pi_{\le N}\varphi-\varphi\|_{L^2} \lesssim t(\varepsilon)^{1-\frac{3\kappa}{8}}\frac{\varepsilon}{2}$, the proof is complete. □

**Lemma 5.10.** *Fix $\omega\in\Omega$. For any $\varphi\ge 0, \varphi\in C^\infty$ and time horizon $T>0$ as well as $\zeta < 2-\frac{d}{2}$, there exists a process $(t,x)\mapsto (U_t\varphi)(x)$ such that $U\varphi\in C([0,T];\mathcal{C}^\zeta)$, with:*

$$U_t\varphi = e^{t\mathcal{H}}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathcal{H}}(U_s\varphi)^2\,\mathrm{d}s.$$

## 6. Stochastic Estimates

This section is devoted to stochastic estimates. In particular, convergence to white noise terms is justified. A crucial ingredient for the proof is the behaviour of multiple discrete stochastic integral. The required results are recalled in the Appendix B.

**Proposition 6.1.** *Fix $\kappa > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a sequence of random functions $\xi_\varepsilon \colon \mathbb{T}^d \to \mathbb{R}$ as in Assumption 1.4. Furthermore, in dimension $d = 2$, for $\lambda > 1$, define*

$$X_{\varepsilon,\lambda} = (-\mathcal{A}_\varepsilon + \lambda)^{-1}\xi_\varepsilon, \qquad \xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda} = \xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda} - c_\varepsilon,$$

*where*

$$c_\varepsilon = \sum_{k\in\mathbb{Z}^2} \frac{\widehat{\chi}^2(\varepsilon k)\widehat{\chi}_Q(\varepsilon k)}{-\vartheta_\varepsilon(k) + \lambda}, \quad \text{with} \quad |c_\varepsilon| \lesssim \log\frac{1}{\varepsilon}.$$

*For any $\zeta \in [0, 1]$*

$$\sup_{\varepsilon\in(0,1/2)} \varepsilon^{\frac{d}{2}\zeta}\mathbb{E}\big[\|\xi_\varepsilon\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)-\kappa}} + \|\mathcal{P}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)+2-\kappa}} + \varepsilon^{-2}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)-\kappa}}\big] < \infty. \tag{46}$$

*Similarly,*

$$\sup_{\omega\in\Omega,\varepsilon\in(0,1/2)} \varepsilon^{\frac{d}{2}}\|\xi_\varepsilon(\omega)\|_{L^\infty} + \varepsilon^{-2+\frac{d}{2}}\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}(\omega)\|_{L^\infty} < \infty. \tag{47}$$

*Additionaly, in dimension $d = 2$*

$$\sup_{\varepsilon\in(0,1/2)} \mathbb{E}\Big[\|\xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda}\|_{\mathcal{C}^{-\kappa}}\Big] < \infty. \tag{48}$$

*Moreover, there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, supporting space white noise $\xi$ on $\mathbb{T}^d$, and a sequence of random functions $\overline{\xi}_\varepsilon \colon \mathbb{T}^d \to \mathbb{R}$ such that*

$$\xi_\varepsilon = \overline{\xi}_\varepsilon, \qquad \text{in distribution,}$$

*such that almost surely*

$$\overline{\xi}_\varepsilon \to \xi \quad \text{in} \quad \mathcal{C}^{-\frac{d}{2}-\kappa}, \qquad \mathcal{P}_\varepsilon\overline{X}_{\varepsilon,\lambda} = \mathcal{P}_\varepsilon(-\mathcal{A}_\varepsilon + \lambda)^{-1}\overline{\xi}_\varepsilon \to (-\Delta + \lambda)^{-1}\xi \quad \text{in} \quad \mathcal{C}^{-\frac{d}{2}+2-\kappa} =: X_\lambda.$$

*In dimension $d = 2$, there exists a random distribution $\xi \diamond X_\lambda$ such that*

$$\xi_\varepsilon \diamond \Pi^2_\varepsilon X_{\varepsilon,\lambda} \to \xi \diamond X_\lambda \quad \text{in} \quad \mathcal{C}^{-\kappa}.$$

*Proof.* The proof is divided into several steps. In the first step, we address (46) and (47). In the second step, we justify the choice of renormalization constant $c_\varepsilon$ and show (48). In the third step the convergence of random variables is established.

*Step 1.* By elliptic Schauder estimates of Lemma 4.15, to show (46) it is sufficient to prove that

$$\sup_{\varepsilon\in(0,1/2)} \varepsilon^{\frac{d}{2}\zeta}\mathbb{E}\|\xi_\varepsilon\|_{\mathcal{C}^{-\frac{d}{2}(1-\zeta)-\kappa}} < \infty.$$

By Besov embedding, up to modifying the value of $\kappa$, for any $p \in [2, \infty)$ this inequality follows if one can show that

$$\sup_{\varepsilon\in(0,1/2)} \varepsilon^{\frac{d}{2}\zeta}\mathbb{E}\|\xi_\varepsilon\|_{B^{-\frac{d}{2}(1-\zeta)-\kappa}_{p,p}} < \infty. \tag{49}$$

Fix $\zeta = 1$. In view of Assumption 1.4, and by discrete Burkholder-Davis-Gundy inequality, and Jensen's inequality one finds that for $p \geq 2$

$$\int_{\mathbb{T}^d} \mathbb{E}[|\Delta_j\varepsilon^{-\frac{d}{2}}s_\varepsilon|^p(x)]\,\mathrm{d}x \lesssim \int_{\mathbb{T}^d}\Bigg(\sum_{z\in\mathbb{Z}^d_\varepsilon}\varepsilon^d|\Delta_j\chi_{Q_\varepsilon}|^2(z{+}x)\Bigg)^{p/2}\mathrm{d}x.$$

$$\leq \int_{\mathbb{T}^d}\bigg(\int_{\mathbb{T}^d}\mathrm{d}z\,|K_j(x{+}z)|^2\bigg)^{p/2}\mathrm{d}x \lesssim \|K_j\|^p_{L^2} \lesssim 2^{j\frac{dp}{2}}.$$

For $\zeta = 0$, since $s_\varepsilon(x) \in (-2.2)$,

$$\int_{\mathbb{T}^d}\mathbb{E}[|\Delta_j\varepsilon^{-\frac{d}{2}}s_\varepsilon|^p(x)]\,\mathrm{d}x \lesssim \|\varepsilon^{-d/2}s_\varepsilon\|^p_\infty \lesssim \varepsilon^{-pd/2}.$$

The inequality (49), and therefore (46), follows by interpolation.

We turn our attention to (47). The bound for $\sup_{\omega\in\Omega,\varepsilon\in(0,1/2)}\varepsilon^{\frac{d}{2}}\|\xi_\varepsilon(\omega)\|_{L^\infty}$ is a direct consequence of Assumption 1.4. As for $\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}$, it is sufficient to notice that

$$
\begin{aligned}
(50)\quad \|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty(\mathbb{T}^d)} &= \|\mathcal{F}^{-1}_{\mathbb{T}^d}[(1-\urcorner)(\varepsilon\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)\widehat{\xi}_\varepsilon(\cdot)]\|_{L^\infty(\mathbb{T}^d)}\\
&\le \|\mathcal{F}^{-1}_{\mathbb{T}^d}[(1-\urcorner)(\varepsilon\cdot)(-\vartheta_\varepsilon+\lambda)^{-1}(\cdot)]\|_{L^1(\mathbb{T}^d)}\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)}\\
&\lesssim \varepsilon^2\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(1-\urcorner)(\varepsilon\cdot)(-\widehat{\chi}^2+1+\lambda)^{-1}(\varepsilon\cdot)]\|_{L^1(\mathbb{R}^d)}\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)}
\end{aligned}
$$

where we applied the Poisson summation formula of Lemma 4.1. Notice that

$$
\begin{aligned}
&\|\mathcal{F}^{-1}_{\mathbb{R}^d}[(1-\urcorner)(\varepsilon\cdot)(-\widehat{\chi}^2+1+\lambda)^{-1}(\varepsilon\cdot)]\|_{L^1(\mathbb{R}^d)}\\
&\quad\le \|\mathcal{F}^{-1}_{\mathbb{R}^d}\Big[\frac{1-\urcorner(\varepsilon\cdot)}{1+\lambda}+(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\lambda}-\frac{1}{1+\lambda}\Big](\varepsilon\cdot)\Big]\|_{L^1(\mathbb{R}^d)}\\
&\quad\le \|\mathcal{F}^{-1}_{\mathbb{R}^d}\Big[\frac{1-\urcorner(\varepsilon\cdot)}{1+\lambda}\Big]\|_{L^1(\mathbb{R}^d)}+\|\mathcal{F}^{-1}_{\mathbb{R}^d}\Big[(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\lambda}-\frac{1}{1+\lambda}\Big](\varepsilon\cdot)\Big]\|_{L^1(\mathbb{R}^d)}
\end{aligned}
$$

The first summand is bounded in $L^1(\mathbb{R}^d)$ uniformly over $\varepsilon$. As for the second observe that

$$
\begin{aligned}
&\|\mathcal{F}^{-1}_{\mathbb{R}^d}\Big[(1-\urcorner)(\varepsilon\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+\lambda}-\frac{1}{1+\lambda}\Big](\varepsilon\cdot)\Big]\|_{L^1(\mathbb{R}^d)}\\
&\qquad\le \sup_{x\in\mathbb{R}^d}(1+|x^d|)\Big|\int_{\mathbb{R}^d}\mathrm{d}k e^{2\pi\iota\langle x,k\rangle}\frac{1}{-\widehat{\chi}^2(k)+1+\lambda}-\frac{1}{1+\lambda}\Big|\\
&\qquad\lesssim \Big\|\frac{1}{-\widehat{\chi}^2(k)+1+\lambda}-\frac{1}{1+\lambda}\Big\|_{L^1(\mathbb{R}^d)}+\Big\|D^2\Big(\frac{1}{-\widehat{\chi}^2(k)+1+\lambda}-\frac{1}{1+\lambda}\Big)\Big\|_{L^1(\mathbb{R}^d)},
\end{aligned}
$$

and both terms are bounded by Lemma 4.6. Combining last two observations with (50) leads to

$$
\|\mathcal{Q}_\varepsilon X_{\varepsilon,\lambda}\|_{L^\infty(\mathbb{T}^d)}\lesssim\varepsilon^2\|\xi_\varepsilon\|_{L^\infty(\mathbb{T}^d)}.
$$

*Step 2.* In this step the choice of the renormalization constant is justified. Then (48) is shown. Define$\psi_0(k_1,k_2)$ and $\widehat{\xi}_\varepsilon(k)$ as

$$
\psi_0(k_1,k_2):=\sum_{|i-j|\le 1}\varrho_i(k_1)\varrho_j(k_2),\qquad \widehat{\xi}_\varepsilon(k):=\mathcal{F}_{\mathbb{T}^d}\xi_\varepsilon(k).
$$

Then

$$
\begin{aligned}
\mathbb{E}\big[\widehat{\xi}_\varepsilon(k_1)\widehat{\xi}_\varepsilon(k_2)\big] &= \int_{(\mathbb{T}^2)^2}e^{-2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\chi_{Q_\varepsilon(x_1)}(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= \int_{\mathbb{T}^2}e^{-2\pi\iota(k_1+k_2)\cdot x_1}\widehat{\chi}_Q(\varepsilon k_2)\,\mathrm{d}x_1=\widehat{\chi}_Q(\varepsilon k_1)1_{\{k_1+k_2=0\}}.
\end{aligned}
$$

To compute the renormalization constant observe that

$$
\begin{aligned}
c_{\varepsilon,\lambda}=\mathbb{E}\big[\xi_\varepsilon\odot\Pi^2_\varepsilon X_{\varepsilon,\lambda}(x)\big] &= \int_{(\mathbb{Z}^2)^2}e^{2\pi\iota(k_1+k_2)\cdot x}\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\mathbb{E}\big[\widehat{\xi}_\varepsilon(k_1)\widehat{\xi}_\varepsilon(k_2)\big]\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&= \int_{\mathbb{Z}^2}\frac{\widehat{\chi}^2(\varepsilon k)\widehat{\chi}_Q(\varepsilon k)}{-\vartheta_\varepsilon(k)+\lambda}\,\mathrm{d}k.
\end{aligned}
$$

The same calculation shows that $c_\varepsilon=\mathbb{E}\big[\xi_\varepsilon X_{\varepsilon,\lambda}\big]$. The bound $c_\varepsilon\le log(1/ve)$ can be deduced in a similar way.

We turn our attention to (48). Even though dimension $d = 2$ is fixed, it is left as a parameter for reader's convenience. As before, for $p \geq 2$, consider

$$\mathbb{E}\|\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon\|^p_{B^\alpha_{p,p}} = \sum_{j\geq -1} 2^{\alpha jp}\mathbb{E}\|\Delta_j(\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon 1_{j=-1})\|^p_{L^p(\mathbb{T}^d)} = \sum_{j\geq -1} 2^{\alpha jp}\int_{\mathbb{T}^d}\mathbb{E}|\Delta_j(\xi_\varepsilon \odot X_{\varepsilon,\lambda} - c_\varepsilon 1_{j=-1})|^p(x)\,\mathrm{d}x. \tag{51}$$

It is convenient to introduce the notation:

$$\mathcal{K}^\varepsilon_m(x) = \mathcal{F}^{-1}_{\mathbb{T}^2}\left(\varrho_m(\cdot)\frac{\widehat{\chi}^2(\varepsilon\cdot)}{-\vartheta_\varepsilon(\cdot)+\lambda}\right)(x).$$

Then the integrand in (51) can be written as

$$\begin{aligned}&\mathbb{E}\big[|\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x) - c_\varepsilon 1_{\{j=-1\}}|^p\big] = \mathbb{E}\big[|\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x) - \mathbb{E}\Delta_j(\xi_\varepsilon \odot \Pi^2_\varepsilon X_{\varepsilon,\lambda})(x)|^p\big] = \\ &\qquad \mathbb{E}\left|\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}\left(\int_{(\mathbb{T}^2)^2}K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\xi_\varepsilon(z_1)\diamond\xi_\varepsilon(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right)\mathrm{d}y\right|^p,\end{aligned} \tag{52}$$

where, conveniently,

$$\xi_\varepsilon(z_1)\diamond\xi_\varepsilon(z_2) = \xi_\varepsilon(z_1)\xi_\varepsilon(z_2) - \mathbb{E}\big[\xi_\varepsilon(z_1)\xi_\varepsilon(z_2)\big].$$

Now write (52) as a discrete stochastic integral and apply Lemma B.2 to obtain Bad formatting, don't know what to do with it

$$\begin{aligned}&\mathbb{E}\left|\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\left(\int_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)}K_l(y-z_1)\mathcal{K}_m(y-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right)\right. \\ &\qquad\qquad \left.\times\,\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2)\,\mathrm{d}y\right|^p \\ &\lesssim \left[\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\varepsilon^{2d}\left|\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}\left(\fint_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)}K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right)\mathrm{d}y\right|^2\right]^{p/2} \\ &= \left[\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\varepsilon^{2d}\left|\fint_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)}\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}K_l(y-z_1)\mathcal{K}_m(y-z_2)\,\mathrm{d}y\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^2\right]^{p/2} \\ &\leq \left[\int_{(\mathbb{T}^2)^2}\left|\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\,\mathrm{d}y\right|^2\mathrm{d}z_1\,\mathrm{d}z_2\right]^{p/2},\end{aligned}$$

where the last step is an application of Jensen's inequality. Now, via Parseval's Theorem, the latter is bounded by

$$\begin{aligned}&\left[\int_{(\mathbb{Z}^2)^2}\left|\int_{\mathbb{T}^2}K_j(x-y)\sum_{|l-m|\leq 1}e^{2\pi\iota k_1\cdot y}\varrho_l(k_1)e^{2\pi\iota k_2\cdot y}\varrho_m(k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\,\mathrm{d}y\right|^2\mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2} \\ &\qquad = \left[\int_{(\mathbb{Z}^2)^2}\left|e^{2\pi\iota(k_1+k_2)\cdot x}\varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}\right|^2\mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2}.\end{aligned}$$

By Lemma 4.6 and Lemma 4.5

$$\frac{\widehat{\chi}^2(\varepsilon k)}{-\vartheta_\varepsilon(k)+\lambda} \lesssim \frac{\widehat{\chi}^2(\varepsilon k)}{|k|^2+\lambda}1_{\{|k\lesssim\varepsilon^{-1}|\}} + \frac{|k|^{-3}}{\lambda}1_{\{|k|\gtrsim\varepsilon^{-1}\}} \lesssim \frac{1}{\lambda+|k|^2}.$$

Finally, taking into account the supports of the functions,

$$\left[\int_{(\mathbb{Z}^2)^2}\left|\varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{1}{1+|k_2|^2}\right|^2 \mathrm{d}k_1\,\mathrm{d}k_2\right]^{p/2} \lesssim \left[2^{j2d}2^{-4j}\right]^{p/2} \le 1,$$

which provides a bound of the required order.

*Step 3.* The discussion so far established tightness of the sequences of random variables

$$\xi_\varepsilon \in \mathcal{C}^{-\frac{d}{2}-\kappa}, \qquad \xi_\varepsilon \diamond X_{\varepsilon,\lambda} \in \mathcal{C}^{-\kappa}, \qquad \forall \kappa > 0.$$

The next step is to show that the limiting points are unique. In particular, in view of Lemma **??**, this would apply weak convergence of $X_{\varepsilon,\lambda}$.

Convergence of $\xi_\varepsilon$ to space time white noise $\xi$ is an instance of central limit theorem (notice the normalization of variance in Assumption 1.4). We therefore focus our attention on the more involved Wick product $\xi_\varepsilon \diamond X_{\varepsilon,\lambda}$.

For fixed $\varphi \in \mathcal{S}(\mathbb{T}^2)$

$$\begin{aligned}
&\langle \varphi, \xi_\varepsilon \diamond X_{\varepsilon,\lambda}\rangle \\
&= \int_{\mathbb{T}^2}\varphi(y)\sum_{|l-m|\le 1}\sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2}\left(\int_{Q_\varepsilon(x_1)\times Q_\varepsilon(x_2)} K_l(y-z_1)\mathcal{K}^\varepsilon_m(y-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right)\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2)\,\mathrm{d}y \\
&\qquad = \sum_{x_1,x_2\in\mathbb{Z}^2_\varepsilon}\Big\langle\varphi(\cdot), \sum_{|l-m|\le 1}\Pi_\varepsilon K_l(\cdot-x_1)\Pi_\varepsilon\mathcal{K}^\varepsilon_m(\cdot-x_2)\Big\rangle\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2).
\end{aligned}$$

Consider a map $L_\varepsilon : \left(\mathbb{Z}^2_\varepsilon\right)^2 \to \mathbb{R}$ defined by

$$L_\varepsilon(x_1,x_2) := \langle\varphi(\cdot), \sum_{|l-m|\le 1}\Pi^Q_\varepsilon K_l(\cdot-x_1)\Pi^Q_\varepsilon\mathcal{K}_m(\cdot-x_2)\rangle 1_{\{(x_1,x_2)\in\mathbb{T}^2\times\mathbb{T}^2\}}.$$

This definition naturally extends to $\varepsilon = 0$, where $L$ maps $(\mathbb{R}^2)^2$ to $\mathbb{R}$.

The goal is to show that

$$\sum_{(x_1,x_2)\in(\mathbb{Z}^2_\varepsilon)^2} L_\varepsilon(x_1,x_2)\xi_\varepsilon(x_1)\diamond\xi_\varepsilon(x_2) \to \int_{(\mathbb{R}^2)^2} L(x_1,x_2)\xi(\,\mathrm{d}x_1)\diamond\xi(\,\mathrm{d}x_2), \tag{53}$$

where convergence holds in distribution and the limit is interpreted as an iterated stochastic integral in the second Wiener-Itô chaos. It is sufficient to verify the assumptions of Lemma B.3. The convergence follows if we verify the assumptions of Lemma B.3. That is, we have to show that there exists a $g \in L^2((\mathbb{R}^2)^2)$ such that:

$$\sup_{\varepsilon\in(0,1)}|1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\mathbb{Z}^2_\varepsilon)^2}L_\varepsilon| \le g, \qquad \lim_{\varepsilon\to 0}\|1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\varepsilon\mathbb{Z}^2)^2}L_\varepsilon - \mathcal{F}_{(\mathbb{R}^2)^2}L\| = 0$$

We calculate

$$\begin{aligned}
&1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\varepsilon\mathbb{Z}^2)^2}L_\varepsilon(k_1,k_2) \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2)\int_{(\mathbb{Z}^2_\varepsilon\cap\mathbb{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\langle\varphi(\cdot), \sum_{|l-m|\le 1}\Pi^Q_\varepsilon K_l(\cdot-x_1)\Pi^Q_\varepsilon\mathcal{K}_m(\cdot-x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2)\int_{(\mathbb{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\langle\varphi(\cdot), \sum_{|l-m|\le 1}K_l(\cdot-x_1)\mathcal{K}_m(\cdot-x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2)\int_{\mathbb{T}^2}\varphi(y)e^{2\pi\iota(k_1+k_2)\cdot y}\sum_{|l-m|\le 1}\varrho_l(-k_1)\varrho_m(-k_2)\frac{\widehat{\chi}^2(-\varepsilon k_2)}{-\vartheta_\varepsilon(-k_2)+\lambda}\,\mathrm{d}y \\
&= 1_{(\varepsilon^{-1}\mathbb{T}^2)^2}(k_1,k_2)(\mathcal{F}_{\mathbb{T}^2}\varphi)(k_1+k_2)\sum_{|l-m|\le 1}\varrho_l(k_1)\varrho_m(k_2)\frac{\widehat{\chi}^2(\varepsilon k_2)}{-\vartheta_\varepsilon(k_2)+\lambda}.
\end{aligned}$$

Since $\varphi$ is smooth, the latter term is bounded in $L^2$, uniformly over $\varepsilon$. In particular (53) follows. Hence the distribution of any limit point of $\langle \varphi, \xi_\varepsilon \odot X_{\varepsilon,\lambda}\rangle$ is uniquely characterized, and, since $\varphi$, is arbitrary this implies convergence in distribution of $\xi_\varepsilon \odot X_{\varepsilon,\lambda}$. Finally, the result concerning almost sure convergence of a different sequence $\bar{\xi}$ follows by Skorohod representation. □

## Appendix A. The spatial Λ-Fleming-Viot process

The aim of this Appendix is to provide a description of the spatial Lambda-Fleming-Viot model in terms of its generator and to describe how to couple such a description with description of a random static environment. Initially, the process is defined conditional on the realisation of the environment. This is sufficient to provide the proof of Lemma 1.2. A short discussion on how to couple the process with the random environment follows.

Let $M_\lambda(\mathbb{T}\times\mathcal{K})$ be space of measures on $\mathbb{T}\times\mathcal{K}$ with the first marginal equal to Lebesgue measure. The SLFV takes values in the space $\mathcal{M}_\lambda(\mathbb{T}\times\mathcal{K})$, where $\mathcal{K}=\{\mathfrak{a},\mathfrak{A}\}$. An application of a standard disintegration theorem guarantees that for every element of $\mathcal{M}_\lambda$ there exists a density $w:\mathbb{F}^d\to[0,1]$

$$M_t(\mathrm{d}x,\mathrm{d}\kappa) = w(t,x)\delta_0(\mathrm{d}\kappa) + (1-w(t,x))\delta_1(\mathrm{d}\kappa).$$

Density $w$ is defined uniquely up to a set of Lebesgue measure 0. We shall think of $w(t,x)$ as a proportion of individuals of type 0 at $x$ at time $t$. For a more detailed and general discussion see [32] , Section 2.

For any function $w:\mathbb{T}^d\to[0,1]$ and $\varepsilon\in(0,\frac{1}{2}),\mathfrak{u}\in(0,1)$ define the operators $\Theta^+_{x,\varepsilon,\mathfrak{u}}$ and $\Theta^-_{x,\varepsilon,\mathfrak{u}}$ by

$$\begin{aligned}\Theta^{\pm}_{x,\varepsilon,\mathfrak{u}}w(y) &= w(y)1_{\{B^c_\varepsilon(x)\}}(y)+(\mathfrak{u}1_{\{+\}}+(1-\mathfrak{u})w(y))1_{\{B_\varepsilon(x)\}}(y)\\ &= w(y)+\mathfrak{u}(1_{\{+\}}-w(y))1_{\{B_\varepsilon(x)\}}(y).\end{aligned}$$

The operator $\Theta^+_{x,\varepsilon,\mathfrak{u}}$ represent the change of the density of the population of type $\mathfrak{a}$ provided that type $\mathfrak{a}$ has been chosen as a parent, while $\Theta^-_{x,\varepsilon,\mathfrak{u}}$ represent the same change if the parent has type $\mathfrak{A}$. We shall omit the dependence on $\varepsilon$ and $\mathfrak{u}$ whenever it does not lead to confusion. Furthermore, for $\varphi\in L^1(\mathbb{T}^d)$ and $F:\mathbb{R}\to\mathbb{R}$ write $F_\varphi$ for the functional

$$F_\varphi(w) = F\bigg(\int_{\mathbb{T}^d} w(x)\varphi(x)\,\mathrm{d}x\bigg).$$

The spatial Lambda-Fleming-Viot process is specified through generator. Its' domain given by

$$\mathcal{D}(\mathcal{L}) = \Big\{F_\varphi \;\big|\; F\in C_b(\mathbb{R};\mathbb{R}),\varphi\in L^1(\mathbb{T}^d;\mathbb{R})\Big\}.$$

We refer to [32, Section 2] for a discussion of the choice of the domain. In the definition below we write:

$$s_+(x) = \max\{s(x),0\},\qquad s_-(x) = \max\{-s(x),0\}.$$

**Definition A.1.** *For fixed $\varepsilon\in(0,1/2),\mathfrak{u}\in(0,1)$ and measurable $s:\mathbb{T}^d\to(-1,1)$ define the operator $\mathcal{L}=\mathcal{L}(\varepsilon,s,\mathfrak{u})$ by*

$$\begin{aligned}\mathcal{L}(F_\varphi)(w) &:= \mathcal{L}^{\mathrm{neu}}(F_\varphi)(w)+\mathcal{L}^{\mathrm{sel}}(F_\varphi)(w)\\ &:= \mathcal{L}^{\mathrm{neu}}(F_\varphi)(w)+\mathcal{L}^{\mathrm{sel}}_<(F_\varphi)(w)+\mathcal{L}^{\mathrm{sel}}_>(F_\varphi)(w),\end{aligned}$$

*where*

$$\mathcal{L}^{\text{neu}}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} (1-|s(x)|)\Big[\Pi_\varepsilon w[F_\varphi(\Theta_x^+ w)-F_\varphi(w)]+(1-\Pi_\varepsilon w)[F_\varphi(\Theta_x^- w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathcal{L}^{\text{sel}}_{>}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} s_-(x)\Big[\big(\Pi_\varepsilon w\big)^2[F_\varphi(\Theta_x^+ w)-F_\varphi(w)]+\big(1-\big(\Pi_\varepsilon w\big)^2\big)[F_\varphi(\Theta_x^- w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathcal{L}^{\text{sel}}_{<}(F_\varphi)(w) = \int\limits_{\mathbb{T}^d} s_+(x)\Big[\Pi_\varepsilon w(2-\Pi_\varepsilon w)[F_\varphi(\Theta_x^+ w)-F_\varphi(w)]+(1-\Pi_\varepsilon w)^2[F_\varphi(\Theta_x^- w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

*The operator $\mathcal{L}$ is the generator of a Markov process Add refs. Which Refs? , that we call the spatial Λ-Fleming-Viot process with selection.*

With this definition at hand we turn our attention to proof of Lemma 1.2.

*Proof of Lemma 1.2.* Simple algebraic transformations of evaluation of the operator $\mathcal{L}$ on functionals of the form $F_\varphi = \mathrm{Id}_\varphi$ show that the neutral part of the generator takes form

$$\mathcal{L}^{\text{neu}}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} (1-|s(x)|)[(\Pi_\varepsilon w)(\Pi_\varepsilon \varphi) - \Pi_\varepsilon(w\varphi)](x)\,\mathrm{d}x,$$

Analogously, the selective part can be written as

$$\mathcal{L}^{\text{sel}}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} s(x)[\Pi_\varepsilon(w\varphi)-\big(\Pi_\varepsilon w\big)^2\Pi_\varepsilon\varphi](x) + 2s_+(x)[\Pi_\varepsilon w\Pi_\varepsilon\varphi-\Pi_\varepsilon(w\varphi)](x)\,\mathrm{d}x.$$

Adding those two we conclude that

$$\mathcal{L}(\mathrm{Id}_\varphi)(w) = \mathfrak{u}\varepsilon^d \int_{\mathbb{T}^d} [(\Pi_\varepsilon w)(\Pi_\varepsilon\varphi) - \Pi_\varepsilon(w\varphi)](x) + s(x)[(\Pi_\varepsilon w)(\Pi_\varepsilon\varphi) - (\Pi_\varepsilon w)^2\Pi_\varepsilon\varphi](x)\,\mathrm{d}x.$$

This justifiesRefer to the drift in the statement(**??**) To obtain the predictable quadratic variation of the martingale make use of Dynkin's formula, that is

$$\langle M^\varepsilon(\varphi)\rangle_t = \int_0^t \mathcal{L}(\mathrm{Id}_\varphi^2) - 2\big(\mathrm{Id}_\varphi \mathcal{L}(\mathrm{Id}_\varphi)\big)(X_r^\varepsilon)\,\mathrm{d}r.$$

The only quantity which still requires evaluation is the second one.. Once again, it is natural to treat the terms involving $\mathcal{L}^{\text{neu}}$ and $\mathcal{L}^{\text{sel}}$ separately. For the neutral term,

$$\begin{aligned}&\big(\mathcal{L}^{\text{neu}}(\mathrm{Id}_\varphi^2) - 2F_\varphi\mathcal{L}^{\text{neu}}(\mathrm{Id}_\varphi)\big)(w)\\ &\qquad = \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d}(1-|s(x)|)\Big[\Pi_\varepsilon w\big(\Pi_\varepsilon\varphi-\Pi_\varepsilon(w\varphi)\big)^2 + \big(1-\Pi_\varepsilon w\big)\big(\Pi_\varepsilon(w\varphi)\big)^2\Big](x)\,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d}(1-|s(x)|)\Big[\Pi_\varepsilon w\big[\big(\Pi_\varepsilon\varphi\big)^2 - 2\Pi_\varepsilon\varphi(x)\Pi_\varepsilon(w\varphi)\big] + \big[\Pi_\varepsilon(w\varphi)\big]^2\Big](x)\,\mathrm{d}x.$$

Analogous calculations for $\mathcal{L}^{\text{sel}}_{<}$ lead to

$$\begin{aligned}&\big(\mathcal{L}^{\text{sel}}_{<}(\mathrm{Id}_\varphi^2) - 2\mathrm{Id}_\varphi\mathcal{L}^{\text{sel}}_{<}\mathrm{Id}_\varphi\big)(w) =\\ &\qquad = \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d} s_-(x)\Big[(\Pi_\varepsilon w)^2\big[(\Pi_\varepsilon\varphi)^2 - 2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\big] + \big[\Pi_\varepsilon(w\varphi)\big]^2\Big](x)\,\mathrm{d}x.\end{aligned}$$

whereas for $\mathcal{L}^{\text{sel}}_{>}$ they lead to

$$\begin{aligned}&\big(\mathcal{L}^{\text{sel}}_{>}(\mathrm{Id}_\varphi^2) - 2\mathrm{Id}_\varphi\mathcal{L}^{\text{sel}}_{>}\mathrm{Id}_\varphi\big)(w)\\ &\qquad = \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d} s_+(x)\Big[(\Pi_\varepsilon w)(2-\Pi_\varepsilon w)\big(\Pi_\varepsilon\varphi-\big(\Pi_\varepsilon(w\varphi)\big)^2 + \big(1-\Pi_\varepsilon w\big)^2\big(\Pi_\varepsilon w\big)^2\Big](x)\,\mathrm{d}x.\\ &\qquad = \mathfrak{u}^2\varepsilon^{2d}\int_{\mathbb{T}^d} s_+(x)\Big[(\Pi_\varepsilon w)(2-\Pi_\varepsilon w)\big[\big(\Pi_\varepsilon\varphi\big)^2-2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\big]+\big[\Pi_\varepsilon(w\varphi)\big]^2(x)\Big]\,\mathrm{d}x.\end{aligned}$$

Summing neutral and selective terms one obtains

$$
\begin{aligned}
&\mathfrak{u}^2\varepsilon^{2d}\langle \Pi_\varepsilon w, (1-|s|)\Big[\big(\Pi_\varepsilon\varphi\big)^2 - 2\Pi_\varepsilon\varphi\Pi_\varepsilon(w\varphi)\Big]\rangle + \langle \big(\Pi_\varepsilon(w\varphi)\big)^2, (1-|s|)\rangle \\
&\quad + \mathfrak{u}\varepsilon^{2d}\langle (\Pi_\varepsilon w)^2, s_-\Big[(\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\Big] + \langle \big(\Pi_\varepsilon(w\varphi)\big)^2, s_-\rangle \\
&\quad + \mathfrak{u}^2\varepsilon^{2d}\langle \Pi_\varepsilon w, s_+\Big[(2-\Pi_\varepsilon w)\Big((\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\Big)\Big]\rangle + \langle \big(\Pi_\varepsilon(w\varphi)\big)^2, s_+\rangle,
\end{aligned}
$$

which can be written in the form from the statement of the Lemma. Probably not necessary

$$
\begin{aligned}
\mathfrak{u}^2\varepsilon^{2d}\Big\{&\langle (1+s)\Pi_\varepsilon w, (\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\rangle + \langle \big(\Pi_\varepsilon(w\varphi)\big)^2, 1\rangle \\
&- \langle s(\Pi_\varepsilon w)^2, (\Pi_\varepsilon\varphi)^2 - 2(\Pi_\varepsilon\varphi)\big(\Pi_\varepsilon(w\varphi)\big)\rangle\Big\}.
\end{aligned}
$$

□

Now we briefly discuss how to couple the construction of the Markov process with the randomness of the noise.

Continue here...

## Appendix B. Discrete stochastic integral

This appendix is devoted to a brief recollection of results on discrete stochastic integral on lattice. The discussion is based on approach of [22, Section 5], which in turn based on [7]. That paper is concerned with processes on Bravais lattices. The attention here is restricted to $\mathbb{Z}^d$ and $\mathbb{Z}^d_\varepsilon$. All results here play a crucial role in Section 6.

**Definition B.1.** *A sequence of independent random variables $\big(\xi_\varepsilon(z)\big)_{z\in\mathbb{Z}^d_\varepsilon}$ is a discrete approximation to white noise if for every $\varepsilon$ it satisfies*

1. $\mathbb{E}[\xi_\varepsilon(z)] = 0$,
2. $\mathbb{E}[|\xi_\varepsilon(z)|^2] = \varepsilon^{-d}$
3. *for some* $p_\xi$ $\sup_{z\in\mathbb{Z}^d_\varepsilon}\mathbb{E}[|\xi_\varepsilon|^{p_\xi}] \le \varepsilon^{-dp_\xi/2}$

The following lemma (see [22, Lemma 5.1] and [7, Theorem 2.3]) provides a crucial estimate for the discrete stochastic integral.

**Lemma B.2.** *Let $\xi_\varepsilon$ be a discrete approximation to white noise. Fix $n \ge 1$ and $p_\xi \ge 2n$. For $f \in L^2\big((\mathbb{Z}^d_e)^n\big)$ define the discrete stochastic integral by*

$$
\mathcal{J}_n f := \sum_{z_1,\dots,z_n\in\mathbb{Z}^d_\varepsilon} f(z_1,\dots,z_n)\xi(z_1)\diamond\cdots\diamond\xi_{z_n}.
$$

*Then for $2 \le p\ p_\xi/n$*

$$
\|\mathcal{J}_n f\|_{L^p} \le \|f\|_{L^2}.
$$

The next lemma (see [22, Lemma 5.4]) provides a convergence criterion for discrete multiple stochastic integrals to multiple Wiener-Itô integrals.

**Lemma B.3.** *Lee $\xi_\varepsilon$ be a discrete approximation to white noise. Fix $n \ge 1$ and $p_\xi \ge 2n$. Assume that there exists a function $g_k \in L^2\big(((\mathbb{R}^d)^k\big)$ such that for all $\varepsilon < 1/2$*

$$
\sup_{\varepsilon\in(0,1/2)} |1_{(\varepsilon^{-1}\mathbb{T}^d)^k}\mathcal{F}_{(\mathbb{Z}^d_\varepsilon)^k}L_\varepsilon| \le g_k,
$$

*and for all $f_k \in L^2\big(((\mathbb{R}^d)^k\big)$ such that for all $k \le n$*

$$
\lim_{\varepsilon\to 0}\|1_{(\varepsilon^{-1}\mathbb{T}^2)^2}\mathcal{F}_{(\varepsilon\mathbb{Z}^d)^k}L_\varepsilon - \mathcal{F}_{(\mathbb{R}^2)^2}L\| = 0.
$$

*Then if* $\xi(\mathrm{d}z_1)\diamond\ldots\xi(\mathrm{d}z_k)$ *denotes the Wiener-Itô integral agains the Gaussian stochastic measure induced by the white noise* $\xi$ *the following convergence holds*

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{n}\mathcal{J}_k f_k^{\varepsilon}=\sum_{k=0}^{n}\int_{(\mathbb{T}^d)^k} f_k(z_1,\ldots,z_k)\xi(\mathrm{d}z_1)\diamond\ldots\xi(\mathrm{d}z_k)$$

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